Existence, stability and instability
for Einstein-scalar field
Lichnerowicz equations

by

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The case $a \geq 0$. Unpublished result.
Given $\Psi$ scalar field, and $V(\Psi)$ a potential, Einstein-scalar field equations are written as:

$$G_{ij} = \nabla_i \Psi \nabla_j \Psi - \frac{1}{2} (\nabla^\alpha \Psi \nabla_\alpha \Psi) \gamma_{ij} - V(\Psi) \gamma_{ij},$$

where $\gamma$ is the spacetime metric, and $G = R_{\gamma\gamma} - \frac{1}{2} S_{\gamma\gamma}$ is the Einstein curvature tensor. In the massive Klein-Gordon field theory,

$$V(\Psi) = \frac{1}{2} m^2 \Psi^2.$$
The constraint equations, using the conformal method, are

\[
\frac{4(n-1)}{n-2} \Delta_g u + h(g, \psi) u = f(\psi, \tau) u^{2* - 1} + \frac{a(\sigma, W, \pi)}{u^{2*+1}}, \quad (1)
\]

\[
\text{div}_g (\mathcal{D}W) = \frac{n-1}{n} u^{2*} \nabla \tau - \pi \nabla \psi, \quad (2)
\]

where \( \Delta_g = -\text{div}_g \nabla \), \( 2^* = 2n/(n-2) \),

\[
h = S_g - |\nabla \psi|^2, \quad a = |\sigma + \mathcal{D}W|^2 + \pi^2, \quad f = 4V(\psi) - \frac{n-1}{n} \tau^2
\]

and \( S_g \) is the scalar curvature of \( g \). Here, \( \psi, \pi \) and \( \tau \) are functions connected to the physics setting (\( \tau \) mean curvature of spacelike hypersurface), \( \sigma \) \( TT \)-tensor, \( W \) vector field, and \( \mathcal{D} \) the conformal Killing operator given by

\[
(\mathcal{D}W)_{ij} = (\nabla_i W)_j + (\nabla_j W)_i - \frac{2}{n} (\text{div}_g W) g_{ij}.
\]

The system (1) – (2) is decoupled in the constant mean curvature setting, namely when \( \tau = C^{te} \).
The free data are \((g, \sigma, \tau, \psi, \pi)\). The determined data are \(u\) and \(W\). They satisfy

\[
\frac{4(n-1)}{n-2} \Delta_g u + h(g, \psi) u = f(\psi, \tau) u^{2* - 1} + \frac{a(\sigma, W, \pi)}{u^{2* + 1}}, \quad (1)
\]

\[
\text{div}_g (D W) = \frac{n-1}{n} u^{2*} \nabla \tau - \pi \nabla \psi. \quad (2)
\]

\((M, g)\) compact, \(\partial M = \emptyset\), \(n \geq 3\). Let \(h, a,\) and \(f\) be arbitrary smooth functions in \(M\). Assume \(a > 0\). Consider

\[\Delta_g u + hu = fu^{2^* - 1} + \frac{a}{u^{2^* + 1}},\]  

\((EL)\)

where \(\Delta_g = -\text{div}_g \nabla\), and \(2^* = \frac{2n}{n-2}\).

**Example:** (Sub and supersolution method, Choquet-Bruhat, Isenberg, Pollack, 2006). Assume \(\Delta_g + h\) is coercive and \(f \leq 0\). Let \(v > 0\) and \(u_0 > 0\) be such that

\[\Delta_g u_0 + hu_0 = v.\]

For \(t > 0\), let \(u_t = tu_0\). We have:

(i) \(u_t\) is a subsolution of \((EL)\) when \(t \ll 1\), and

(ii) \(u_t\) is a supersolution of \((EL)\) when \(t \gg 1\).

Since \(u_t \leq u_{t'}\) when \(t \leq t'\), the sub and supersolution method provides a solution “\(u \in [u_t, u_{t'}]\)” for \((EL)\).
Question: What can we say when $\Delta g + h$ is coercive and either $f$ changes sign or $f$ is everywhere positive, i.e. when $\max_M f > 0$?

Assume $\Delta g + h$ is coercive. Define

$$\|u\|_h^2 = \int_M (|\nabla u|^2 + hu^2) \, dv_g,$$

$u \in H^1$. Let $S(h)$ to be the smallest constant such that

$$\left(\int_M |u|^{2^*} \, dv_g \right)^{2/2^*} \leq S(h)^{2/2^*} \int_M (|\nabla u|^2 + hu^2) \, dv_g$$

for all $u \in H^1$. 
\((M, g)\) be a smooth compact Riemannian manifold, \(n \geq 3\). Let \(h, a,\) and \(f\) be smooth functions in \(M\). Assume that \(\Delta_g + h\) is coercive, that \(a > 0\) in \(M\), and that \(\max_M f > 0\). There exists 
\(C = C(n), C > 0\) depending only on \(n\), such that if 
\[
\|\varphi\|_h^2 \int_M \frac{a}{\varphi^2} dv_g < \frac{C(n)}{(S(h) \max_M |f|)^{n-1}}
\]
and \(\int_M f \varphi^{2*} dv_g > 0\) for some smooth positive function \(\varphi > 0\) in 
\(M\), then the Einstein-scalar field Lichnerowicz equation (EL) possesses a smooth positive solution.

Example: if \(\int_M f dv_g > 0\) then take \(\varphi \equiv 1\) and the condition reads as 
\[
\int_M a dv_g < \frac{C(n, g, h)}{(\max_M |f|)^{n-1}} ,
\]
where \(C(n, g, h) > 0\) depends on \(n, g\) and \(h\).
A perturbation of (EL) is a sequence \((EL_\alpha)_\alpha\) of equations, \(\alpha \in \mathbb{N}\), which are written as

\[
\Delta_g u + h_\alpha u = f_\alpha u^{2^*-1} + \frac{a_\alpha}{u^{2^*+1}} + k_\alpha
\]

\((EL_\alpha)\)

for all \(\alpha\). Here we require that

\[
h_\alpha \to h, \ a_\alpha \to a, \ k_\alpha \to 0
\]

in \(C^0\) as \(\alpha \to +\infty\), and that \(f_\alpha \to f\) in \(C^{1,\eta}\) as \(\alpha \to +\infty\), where \(\eta > \frac{1}{2}\).

If (EL) satisfies the assumption of Theorem 1, any perturbation of (EL) also satisfies the assumptions of Theorem 1.

A sequence \((u_\alpha)_\alpha\) is a sequence of solutions of \((EL_\alpha)_\alpha\) if for any \(\alpha\), \(u_\alpha\) solves \((EL_\alpha)\).
**Definition:** (Elliptic stability) The Einstein-scalar field Lichnerowicz equation (EL) is said to be:

(i) stable if for any perturbation \((EL_\alpha)_\alpha\) of (EL), and any \(H^1\)-bounded sequence \((u_\alpha)_\alpha\) of smooth positive solutions of \((EL_\alpha)_\alpha\), there exists a smooth positive solution \(u\) of (EL) such that, up to a subsequence, \(u_\alpha \to u\) in \(C^{1,\theta}(M)\) for all \(\theta \in (0,1)\), and

(ii) bounded and stable if for any perturbation \((EL_\alpha)_\alpha\) of (EL), and any sequence \((u_\alpha)_\alpha\) of smooth positive solutions of \((EL_\alpha)_\alpha\), the sequence \((u_\alpha)_\alpha\) is bounded in \(H^1\) and there exists a smooth positive solution \(u\) of (EL) such that, up to a subsequence, \(u_\alpha \to u\) in \(C^{1,\theta}(M)\) for all \(\theta \in (0,1)\).
**Remark 1:** Assuming stronger convergences for the $h_\alpha$'s, $f_\alpha$'s, etc., then we get stronger convergences for the $u_\alpha$'s. E.g., if $h_\alpha \to h$, $f_\alpha \to f$, $a_\alpha \to a$ and $k_\alpha \to 0$ in $C^{p,\theta}$, $p \in \mathbb{N}$ and $\theta \in (0, 1)$, then $u_\alpha \to u$ in $C^{p+2,\theta'}$, $\theta' < \theta$.

**Remark 2:** Stability means that if you slightly perturb $h$, $a$, and $f$, and even if you add to the equation a small “background noise” represented by $k$, then, in doing so, you do not create solutions which stand far from a solution of the original equation.

**Remark 3:** Say $(EL)$ is compact if any $H^1$-bounded sequence $(u_\alpha)_\alpha$ of solutions of $(EL)$ does possess a subsequence which converges in $C^2$. Say $(EL)$ is bounded and compact if any sequence $(u_\alpha)_\alpha$ of solutions of $(EL)$ does possess a subsequence which converges in $C^2$. Stability implies compactness. Bounded stability implies bounded compactness.
Let $\mathcal{D} = C^\infty(M)^4$ and $\| \cdot \|_\mathcal{D}$ be given by

$$
\| D \|_\mathcal{D} = \sum_{i=1}^{3} \| f_i \|_{C^{0,1}} + \| f_4 \|_{C^{1,1}}
$$

for all $D = (f_1, f_2, f_3, f_4) \in \mathcal{D}$. For $D = (h, a, k, f)$ in $\mathcal{D}$ consider

$$
\Delta_g u + hu = fu^{2^* - 1} + \frac{a}{u^{2^* + 1}} + k. \quad (EL')
$$

If $D = (h, a, 0, f)$, then $(EL') = (EL)$. Let $\Lambda > 0$, $D = (h, a, k, f)$ in $\mathcal{D}$, and define

$$
S_{D,\Lambda} = \left\{ u \text{ solution of } (EL') \text{ s.t. } \|u\|_{H^1} \leq \Lambda \right\},
$$

and $S_D = \left\{ u \text{ solution of } (EL') \right\}$.

When $D = (h, a, 0, f)$ we recover solutions of $(EL)$. 
For $X, Y \subset C^2$ define

\[
d_{C^2}(X; Y) = \sup_{u \in X} \inf_{v \in Y} \| v - u \|_{C^2}.
\]

By convention, $d_{C^2}(X; \emptyset) = +\infty$ if $X \neq \emptyset$, and $d_{C^2}(\emptyset; Y) = 0$ for all $Y$, including $Y = \emptyset$.

Let $D = (h, a, 0, f)$ be given.

Stability $\iff$ $(EL)$ is compact and

\[
\forall \varepsilon > 0, \ \forall \Lambda > 0, \ \exists \delta > 0 \text{ s.t. } \forall D' = (h', a', k', f') \in D, \langle D' - D \rangle_D < \delta \implies d_{C^2}(S_{D'}, \Lambda; S_D, \Lambda) < \varepsilon.
\]

Bounded stability $\iff$ $(EL)$ is bounded compact and

\[
\forall \varepsilon > 0, \ \exists \delta > 0 \text{ s.t. } \forall D' = (h', a', k', f') \in D, \langle D' - D \rangle_D < \delta \implies d_{C^2}(S_{D'}; S_D) < \varepsilon.
\]
Theorem 2: (Druet-H., Math. Z., 2008) Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 3\), and \(h, a, f \in C^\infty(M)\) be smooth functions in \(M\) with \(a > 0\). Assume \(n = 3, 4, 5\). Then the Einstein-scalar field Lichnerowicz equation

\[
\Delta_g u + hu = fu^{2^*-1} + \frac{a}{u^{2^*}+1}
\]

is stable. The equation is even bounded and stable assuming in addition that \(f > 0\) in \(M\). On the contrary, \((EL)\) is not anymore stable a priori when \(n \geq 6\).
I. Further directions and comments - 1

Assume $a \geq 0$, $f > 0$, and

$$\frac{n^n}{(n - 1)^{n-1}} \left( \int_M a^\frac{n+2}{4n} f^\frac{3n-2}{4n} d\nu_g \right)^{\frac{4n}{n+2}} > \left( \int_M \left( h^+ \right)^\frac{n+2}{4} d\nu_g \right)^{\frac{4n}{n+2}}.$$ 

Then the Einstein-scalar field Lichnerowicz equation (EL) does not possess solutions.

In particular, for any $h$, and any $f > 0$, there exist a positive constante $C = C(n, g, h, f)$ such that if

$$\int_M a^\frac{n+2}{4n} d\nu_g \geq C,$$

then (EL) does not possess solutions.
Proof: Integrating \((EL)\),

\[
\int_M f u^{2^*-1} dv_g + \int_M \frac{a dv_g}{u^{2^*+1}} = \int_M hudv_g.
\]

By Hölder’s inequalities,

\[
\int_M hudv_g \leq \left( \int_M \left( h^+ \right)^{\frac{n+2}{4}} dv_g \right)^{\frac{4}{n+2}} \left( \int_M f u^{2^*-1} dv_g \right)^{\frac{n-2}{n+2}}, \quad \text{and}
\]

\[
\int_M a^{\frac{n+2}{4n}} f^{\frac{3n-2}{4n}} dv_g \leq \left( \int_M f u^{2^*-1} dv_g \right)^{\frac{3n-2}{4n}} \left( \int_M \frac{a dv_g}{u^{2^*+1}} \right)^{\frac{n+2}{4n}}.
\]
\[
X + \left( \int_M a \frac{n+2}{4n} f \frac{3n-2}{4n} d\nu_g \right)^\frac{4n}{n+2} X^{1-n} \leq \left( \int_M \frac{(h^+)^{\frac{n+2}{4}}}{f^{\frac{n-2}{4}}} d\nu_g \right)^\frac{4n}{n+2},
\]

where
\[
X = \left( \int_M a \frac{n+2}{4n} f \frac{3n-2}{4n} d\nu_g \right)^\frac{4n}{n+2}.
\]

This implies
\[
\frac{n^n}{(n-1)^{n-1}} \left( \int_M a \frac{n+2}{4n} f \frac{3n-2}{4n} d\nu_g \right)^\frac{4n}{n+2} \leq \left( \int_M \frac{(h^+)^{\frac{n+2}{4}}}{f^{\frac{n-2}{4}}} d\nu_g \right)^\frac{4n}{n+2}.
\]
II. Further directions and comments - 2

Fix $h$, $a$ and $f$. Assume $\Delta g + h$ is coercive, and $a, f > 0$. Let $t > 0$ and consider

$$\Delta g u + hu = fu^{2^*-1} + \frac{ta}{u^{2^*+1}}.$$  \hspace{1cm} (EL_t)

According to Theorem 1 and the Lemma:

(i) (Theorem 1) for $t \ll 1$, $(EL_t)$ possesses a solution,

(ii) (Lemma 1) for $t \gg 1$, $(EL_t)$ does not possess any solution.

Assuming $n = 3, 4, 5,$

(iii) (Theorem 2) $(EL_t)_t$ is bounded and stable for $t \in [t_0, t_1],$

where $0 < t_0 < t_1.$
Let $\Lambda > 0$. Define
\[
\Omega_\Lambda = \left\{ u \in C^{2,\theta} \text{ s.t. } \| u \|_{C^{2,\theta}} < \Lambda \text{ and } \min_M u > \Lambda^{-1} \right\}.
\]

Fix $t_0 \ll 1$ such that $(EL_{t_0})$ possesses a solution. Fix $t_1 \gg 1$ such that $(EL_{t_1})$ does not possess any solution. Assume $n = 3, 4, 5$.

Define $F_t : \overline{\Omega}_\Lambda \to C^{2,\theta}$ by
\[
F_t u = u - L^{-1} \left( fu^{2^*-1} + \frac{ta}{u^{2^*}+1} \right),
\]
where $L = \Delta g + h$, and $t \in [t_0, t_1]$. By (iii), there exists $\Lambda_0 > 0$ such that $F_t^{-1}(0) \subset \Omega_{\Lambda_0}$ for all $t \in [t_0, t_1]$. Then, by (ii),
\[
\deg(F_{t_0}, \Omega_{\Lambda}, 0) = 0
\]
for all $\Lambda \gg 1$. In particular, assuming that the solutions of the equations are nondegenerate, the solution in Theorem 1 needs to come with another solution.
III. Proof of Theorem 1

We aim in proving:

Let \((M, g)\) be a smooth compact Riemannian manifold, \(n \geq 3\). Let \(h, a, \) and \(f\) be smooth functions in \(M\). Assume that \(\Delta g + h\) is coercive, that \(a > 0\) in \(M\), and that \(\max_M f > 0\). There exists \(C = C(n), C > 0\) depending only on \(n\), such that if

\[
\|\varphi\|_h^2 \int_M \frac{a}{\varphi^{2^*}} d\nu_g < \frac{C(n)}{(S(h) \max_M |f|)^{n-1}}
\]

and \(\int_M f \varphi^{2^*} d\nu_g > 0\) for some smooth positive function \(\varphi > 0\) in \(M\), then the Einstein-scalar field Lichnerowicz equation

\[
\Delta_g u + hu = fu^{2^*-1} + \frac{a}{u^{2^*+1}} \quad (EL)
\]

possesses a smooth positive solution.

Method: approximated equations, mountain pass analysis.
Fix $\epsilon > 0$. Define

$$I^{(1)}(u) = \frac{1}{2} \int_M (|\nabla u|^2 + hu^2) \, dv_g - \frac{1}{2^*} \int_M f(u^*)^{2*} \, dv_g,$$

and

$$I^{(2)}(\epsilon) = \frac{1}{2^*} \int_M \frac{adv_g}{(\epsilon + (u^+)^2)^{2^*/2}},$$

where $u \in H^1$. Let

$$I_\epsilon = I^{(1)} + I^{(2)}.$$

Let $\varphi > 0$ be as in Theorem 1. Assume $\|\varphi\|_h = 1$. The conditions in the theorem read as

$$\int_M \frac{a}{\varphi^{2^*}} \, dv_g < \frac{C(n)}{(S(h) \max_M |f|)^{n-1}}$$

$$\tag{1}$$

and $\int_M f \varphi^{2^*} \, dv_g > 0.$
Let $\Phi, \Psi : \mathbb{R}^+ \to \mathbb{R}$ be the functions given by
\[
\Phi(t) = \frac{1}{2} t^2 - \frac{\max_{\mathcal{M}} |f|}{2^*} S(h)t^{2^*}, \quad \text{and}
\Psi(t) = \frac{1}{2} t^2 + \frac{\max_{\mathcal{M}} |f|}{2^*} S(h)t^{2^*}.
\]

These functions satisfy
\[
\Phi(\|u\|_h) \leq l^{(1)}(u) \leq \Psi(\|u\|_h) \quad (2)
\]
for all $u \in H^1$. Let $t_1 > 0$ be such that $\Phi$ is increasing up to $t_1$ and decreasing after:
\[
t_1 = \left( S(h) \max_{\mathcal{M}} |f| \right)^{-(n-2)/4}.
\]

Let $t_0 > 0$ be given by
\[
t_0 = \sqrt{\frac{1}{2(n-1)} t_1}.
\]
Then
\[ \psi(t_0) \leq \frac{1}{2} \Phi(t_1) \quad (3) \]
and for \( C \ll 1 \) the condition in the theorem translates into
\[ \frac{1}{2^*} \int_M \frac{a}{(t_0 \varphi)^{2^*}} dV_g < \frac{1}{2} \Phi(t_1). \quad (4) \]

Let \( \rho = \Phi(t_1) \). Then, by (3) and (4),
\[ I_{\varepsilon}(t_0 \varphi) < \rho \]
and by
\[ \Phi(\|u\|_h) \leq I^{(1)}(u) \leq \psi(\|u\|_h), \quad (2) \]
we can write that
\[ I_{\varepsilon}(u) \geq \rho \]
for all \( u \) s.t. \( \|u\|_h = t_1 \).
We got that there exists $\rho > 0$ such that

$$I_\epsilon(t_0 \varphi) < \rho$$

and

$$I_\epsilon(u) \geq \rho$$

for all $u$ s.t. $\|u\|_h = t_1$. Also $t_1 > t_0$. Since $\int_M f \varphi^{2*} dv_g > 0$, 

$$I_\epsilon(t \varphi) \to -\infty$$

as $t \to +\infty$.

$\Rightarrow$ We can apply the mountain pass lemma.
Let $t_2 \gg 1$. Define
\[ c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} I_\varepsilon(u) \]
where $\Gamma$ is the set of continuous paths joining $t_0 \varphi$ to $t_2 \varphi$. The MPL provides a Palais-Smale sequence $(u_\varepsilon^k)_k$ such that
\[ I_\varepsilon(u_\varepsilon^k) \to c_\varepsilon \quad \text{and} \quad I_\varepsilon'(u_\varepsilon^k) \to 0 \]
as $k \to +\infty$. The sequence $(u_\varepsilon^k)_k$ is bounded in $H^1$. Up to a subsequence, $u_\varepsilon^k \rightharpoonup u_\varepsilon$ in $H^1$. Then $u_\varepsilon$ satisfies
\[ \Delta_g u_\varepsilon + h u_\varepsilon = f u_\varepsilon^{2^* - 1} + \frac{a u_\varepsilon}{(\varepsilon + u_\varepsilon^2)^{2^*/2} + 1} \]
In particular, $u_\varepsilon$ is positive and smooth.
We can prove that the $c_\varepsilon$’s are bounded independently of $\varepsilon$. In particular the family $(u_\varepsilon)_\varepsilon$ is bounded in $H^1$. Now we can pass to the limit as $\varepsilon \to 0$ because $u_\varepsilon$ will never approach zero. Take $x_\varepsilon \in M$ such that $u_\varepsilon(x_\varepsilon) = \min_M u_\varepsilon$. Then $\Delta_g u_\varepsilon(x_\varepsilon) \leq 0$ and

$$|h(x_\varepsilon)| + |f(x_\varepsilon)| u_\varepsilon(x_\varepsilon)^{2^* - 2} \geq \frac{a(x_\varepsilon)}{(\varepsilon + u_\varepsilon(x_\varepsilon)^2)^{2^*} + 1}.$$ 

This implies that there exists $\delta_0 > 0$ such that

$$\min_M u_\varepsilon \geq \delta_0$$

for all $\varepsilon$. If $u_\varepsilon \rightharpoonup u$ in $H^1$, then $u \geq \delta_0$ and $u$ solves $(EL)$.

\diamond
IV. Proof of the stability part in Theorem 2

We aim in proving:

Let \((M,g)\) be a smooth compact Riemannian manifold of dimension \(n \geq 3\), and \(h, a, f \in C^\infty(M)\) be smooth functions in \(M\) with \(a > 0\). Assume \(n = 3, 4, 5\). Then the Einstein-scalar field Lichnerowicz equation

\[
\Delta_g u + hu = fu^{2^*-1} + \frac{a}{u^{2^*+1}} \quad (EL)
\]

is stable, and even bounded and stable if \(f > 0\) in \(M\).

Method: blow-up analysis, sharp pointwise estimates.
Let \((EL_\alpha)_\alpha\) be a perturbation of \((EL)\). Let also \((u_\alpha)_\alpha\) be a sequence of solutions of \((EL_\alpha)\). Consider

(H1A) \(f > 0\) in \(M\),

(H1B) \((u_\alpha)_\alpha\) is bounded in \(H^1\),

(H2) \(\exists \varepsilon_0 > 0\) s.t. \(u_\alpha \geq \varepsilon_0\) in \(M\) for all \(\alpha\).

We claim that:

**Stability Theorem:** (Druet-H., Math. Z., 2008) Let \(n \leq 5\). Let \((EL_\alpha)_\alpha\) be a perturbation of \((EL)\) and \((u_\alpha)_\alpha\) a sequence of solutions of \((EL_\alpha)_\alpha\). Assume (H1A) or (H1B), and we also assume (H2). Then the sequence \((u_\alpha)_\alpha\) is uniformly bounded in \(C^{1,\theta}\), \(\theta \in (0, 1)\).

By (H2),

\[ |\Delta_g u_\alpha| \leq C u_\alpha^{2* - 1} , \]

where \(C > 0\) does not depend on \(\alpha\).
Proof of stability theorem: By contradiction. We assume that \( \|u_\alpha\|_\infty \to +\infty \) as \( \alpha \to +\infty \). We also assume (H1A) or (H1B), and (H2). Let \((x_\alpha)_\alpha\) and \((\rho_\alpha)_\alpha\) be such that

(i) \(x_\alpha\) is a critical point of \(u_\alpha\) for all \(\alpha\),

(ii) \(\rho_\alpha^{\frac{n-2}{2}} \sup_{B_{x_\alpha}(\rho_\alpha)} u_\alpha \to +\infty\) as \(\alpha \to +\infty\), and

(iii) \(d_g(x_\alpha, x) \frac{n-2}{2} u_\alpha(x) \leq C\) for all \(x \in B_{x_\alpha}(\rho_\alpha)\) and all \(\alpha\).

Then :

Main Estimate: Assume (i) – (iii). Then we have that \(\rho_\alpha \to 0\), \(\frac{n-2}{2} u_\alpha(x_\alpha) \to +\infty\), and

\[
 u_\alpha(x_\alpha) \rho_\alpha^{n-2} u_\alpha \left( \exp_{x_\alpha}(\rho_\alpha x) \right) \to \frac{\lambda}{|x|^{n-2}} + H(x)
\]

in \(C^2_{loc}(B_0(1)\setminus\{0\})\) as \(\alpha \to +\infty\), where \(\lambda > 0\) and \(H\) is a harmonic function in \(B_0(1)\) which satisfies that \(H(0) = 0\).
There exist $C > 0$, a sequence $(N_\alpha)_{\alpha}$ of integers, and for any $\alpha$, critical points $x_{1,\alpha}, \ldots, x_{N_\alpha,\alpha}$ of $u_\alpha$ such that

$$\left( \min_{i=1,\ldots,N_\alpha} d_g(x_{i,\alpha}, x) \right)^{\frac{n-2}{2}} u_\alpha(x) \leq C$$

for all $x \in M$ and all $\alpha$. We have $N_\alpha \geq 2$. Define

$$d_\alpha = \min_{1 \leq i < j \leq N_\alpha} d_g(x_{i,\alpha}, x_{j,\alpha})$$

and let the $x_{i,\alpha}$'s be such that $d_\alpha = d_g(x_{1,\alpha}, x_{2,\alpha})$. We have $d_\alpha \to 0$ as $\alpha \to +\infty$. Moreover,

$$d_\alpha^{\frac{n-2}{2}} u_\alpha(x_{1,\alpha}) \to +\infty$$

as $\alpha \to +\infty$. 
Define $\tilde{u}_\alpha$ by

$$\tilde{u}_\alpha(x) = d_\alpha^{\frac{n-2}{2}} u_\alpha \left( \exp_{x_1, \alpha}(d_\alpha x) \right),$$

where $x \in \mathbb{R}^n$. Let $\tilde{v}_\alpha = \tilde{u}_\alpha(0)\tilde{u}_\alpha$. Then

$$|\Delta \tilde{g}_\alpha \tilde{v}_\alpha| \leq \frac{C}{\tilde{u}_\alpha(0)^{2*-2}} \tilde{v}_\alpha^{2*-1},$$

(3)

where $\tilde{g}_\alpha \to \delta$ as $\alpha \to +\infty$. Because of

$$d_\alpha^{\frac{n-2}{2}} u_\alpha(x_1, \alpha) \to +\infty,$$

(2)

$\tilde{u}_\alpha(0) \to +\infty$ as $\alpha \to +\infty$. Independently, by elliptic theory, for any $R > 0$,

$$\tilde{v}_\alpha \to G \text{ in } C^1_{loc} (B_0(R) \setminus \{\tilde{x}_i\}_{i=1,...,p})$$

as $\alpha \to +\infty$, where, because of (3), $G$ is nonnegative and harmonic in $B_0(R) \setminus \{\tilde{x}_i\}_{i=1,...,p}$. 
Then,

\[ G(x) = \sum_{i=1}^{p} \frac{\lambda_i}{|x - \tilde{x}_i|^{n-2}} + H(x), \]

where \( \lambda_i > 0 \) and \( H \) is harmonic without singularities. In particular, in a neighbourhood of 0,

\[ G(x) = \frac{\lambda_1}{|x|^{n-2}} + \tilde{H}(x). \]

By

\[ \left( \min_{i=1,\ldots,N_\alpha} d_g(x_i,\alpha, x) \right)^{\frac{n-2}{2}} u_\alpha(x) \leq C \] \hspace{1cm} (1)

\[ d_\alpha^{\frac{n-2}{2}} u_\alpha(x_{1,\alpha}) \to +\infty \] \hspace{1cm} (2)

we can apply the main estimate with \( x_\alpha = x_{1,\alpha} \) and \( \rho_\alpha = \frac{d_\alpha}{10} \). In particular, \( \tilde{H}(0) = 0. \)
However,

\[ G(x) = \frac{\lambda_1}{|x|^{n-2}} + \frac{\lambda_2}{|x - \tilde{x}_2|^{n-2}} + \hat{H}(x) \]

\[ \geq 0 \]

and

\[ \tilde{H}(x) = \frac{\lambda_2}{|x - \tilde{x}_2|^{n-2}} + \hat{H}(x) . \]

By the maximum principle,

\[ \hat{H}(0) \geq \min_{\partial B_0(R)} \hat{H} \]

and we get that

\[ \tilde{H}(0) \geq \frac{\lambda_2}{|\tilde{x}_2|^{n-2}} - \frac{\lambda_1}{R^{n-2}} - \frac{\lambda_2}{(R - |\tilde{x}_2|)^{n-2}} . \]

By construction, \(|\tilde{x}_2| = 1\). Choosing \(R \gg 1\) sufficiently large, \(\tilde{H}(0) > 0\). A contradiction.
It remains to prove the stability part in theorem 2. We introduced

(H1A) \( f > 0 \) in \( M \),

(H1B) \( (u_\alpha)_\alpha \) is bounded in \( H^1 \),

(H2) \( \exists \varepsilon_0 > 0 \) s.t. \( u_\alpha \geq \varepsilon_0 \) in \( M \) for all \( \alpha \),

and we proved that

\[(H1A) \text{ or } (H1B), \text{ and } (H2) \Rightarrow C^{1,\theta} - \text{convergences}\]

for the \( u_\alpha \)'s solutions of perturbations of \( (EL) \). Let \( (EL_\alpha)_\alpha \) be any perturbation of \( (EL) \), and \( (u_\alpha)_\alpha \) be any sequence of solution of \( (EL_\alpha)_\alpha \). It suffices to prove (H2). Let \( x_\alpha \) be such that

\[ u_\alpha(x_\alpha) = \min u_\alpha. \]

Then \( \Delta_g u_\alpha(x_\alpha) \leq 0 \) and we get that

\[ h_\alpha(x_\alpha) \geq \frac{1}{u_\alpha(x_\alpha)} \left( \frac{a_\alpha(x_\alpha)}{u_\alpha(x_\alpha)^{2x+1}} + k_\alpha(x_\alpha) \right) + f_\alpha(x_\alpha)u_\alpha(x_\alpha)^{2x-2}. \]

In particular, \( u_\alpha \geq \varepsilon_0 > 0 \) and (H2) is satisfied. We can apply the stability theorem. This proves the stability part of Theorem 2.
V. Proof of the instability part in Theorem 2

We aim in proving:

When \( n \geq 6 \) the Einstein-scalar field Lichnerowicz equation

\[
\Delta_g u + hu = fu^{2*-1} + \frac{a}{u^{2*+1}}
\]  \hspace{1cm} (EL)

is not a priori stable.

Method: explicit constructions of examples.
A first construction.

**Lemma 2:** (Druet-H., Math. Z., 2008) Let \((S^n, g_0)\) be the unit sphere, \(n \geq 7\). Let \(x_0 \in S^n\). Let \(a\) and \(u_0\) be smooth positive functions such that

\[
\Delta g_0 u_0 + \frac{n(n-2)}{4} u_0 = \frac{n(n-2)}{4} u_0^{2^*-1} + \frac{a}{u_0^{2^*+1}}.
\]

There exist sequences \((h_\alpha)_{\alpha}\) and \((\Phi_\alpha)_{\alpha}\) such that \(h_\alpha \to \frac{n(n-2)}{4}\) in \(C^0(S^n)\), \(\max_M \Phi_\alpha \to +\infty\) and \(\Phi_\alpha \to 0\) in \(C^2_{loc}(S^n \setminus \{x_0\})\) as \(\alpha \to +\infty\). In addition

\[
\Delta g_0 u_\alpha + h_\alpha u_\alpha = \frac{n(n-2)}{4} u_\alpha^{2^*-1} + \frac{a}{u_\alpha^{2^*+1}}
\]

for all \(\alpha\), where \(u_\alpha = u_0 + \Phi_\alpha\).
Proof of Lemma 2: Let $\varphi_\alpha$ be given by

$$\varphi_\alpha(x) = \left( \frac{\sqrt{\beta^2_\alpha - 1}}{\beta_\alpha - \cos d_{g_0}(x_0, x)} \right)^{\frac{n-2}{2}},$$

where $\beta_\alpha > 1$ for all $\alpha$ and $\beta_\alpha \to 1$ as $\alpha \to +\infty$. The $\varphi_\alpha$'s satisfy

$$\Delta_{g_0} \varphi_\alpha + \frac{n(n-2)}{4} \varphi_\alpha = \frac{n(n-2)}{4} \varphi_\alpha^{2^*-1}.$$

Let

$$u_\alpha = u_0 + \varphi_\alpha + \psi_\alpha,$$

where $\psi_\alpha$ is such that

$$\Delta_{g_0} u_0 + \Delta_{g_0} \varphi_\alpha + \Delta_{g_0} \psi_\alpha$$

$$= \frac{n(n-2)}{4} (u_0 + \varphi_\alpha)^{2^*-1} - \left( \frac{n(n-2)}{4} + \varepsilon_\alpha \right) (u_0 + \varphi_\alpha)$$

$$+ \frac{a}{(u_0 + \varphi_\alpha)^{2^*+1}}.$$

We have $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$. 
For any sequence \((x_\alpha)_\alpha\) of points in \(S^n\),

\[
|\psi_\alpha(x_\alpha)| = o \left( \left( \frac{(\beta_\alpha - 1)^{\frac{(n-2)}{2(n-4)}}}{(\beta_\alpha - 1) + d_{g_0}(x_0, x_\alpha)^2} \right)^{\frac{n-4}{2}} \right) + o(1) . \quad (1)
\]

Thanks to (1),

\[
\frac{\psi_\alpha}{u_\alpha} \to 0 \quad \text{and} \quad u_\alpha^{2^*-3} \psi_\alpha \to 0 \quad (2)
\]

in \(C^0(S^n)\) as \(\alpha \to +\infty\). For instance, either \(\psi_\alpha(x_\alpha) \to 0\) and \(\psi_\alpha(x_\alpha)/u_\alpha(x_\alpha) \to 0\), or \(\psi_\alpha(x_\alpha) \not\to 0\). In that case, because of (1), \(d_{g_0}(x_0, x_\alpha) \to 0\). Then, \(\psi_\alpha(x_\alpha)/u_\alpha(x_\alpha) \to 0\) since

\[
\frac{\psi_\alpha(x_\alpha)}{\varphi_\alpha(x_\alpha)} \leq C^{te} \left( (\beta_\alpha - 1) + d_{g_0}(x_0, x_\alpha)^2 \right) .
\]
Let $h_\alpha$ be such that
\[
\Delta g_0 u_\alpha + h_\alpha u_\alpha = \frac{n(n-2)}{4} u_\alpha^{2*-1} + \frac{a}{u_\alpha^{2*+1}}
\]
for all $\alpha$. Write $\Delta g_0 u_\alpha = \Delta g_0 u_0 + \Delta g_0 \varphi_\alpha + \Delta g_0 \psi_\alpha$. By the equation satisfied by $\psi_\alpha$,
\[
\left(h_\alpha - \frac{n(n-2)}{4}\right) u_\alpha = O \left(u_\alpha^{2*-2}\psi_\alpha\right) + O \left(\psi_\alpha\right) + \varepsilon_\alpha u_\alpha.
\]
Divide by $u_\alpha$, and conclude thanks to
\[
\frac{\psi_\alpha}{u_\alpha} \to 0 \quad \text{and} \quad u_\alpha^{2*-3}\psi_\alpha \to 0 \quad (2)
\]
that $h_\alpha \to \frac{n(n-2)}{4}$ in $C^0$ as $\alpha \to +\infty$. This proves Lemma 2. ♦
Say that \((M, g)\) has a conformally flat pole at \(x_0\) if \(g\) is conformally flat around \(x_0\). Thanks to Lemma 2 we get:

**Lemma 3:** (Druet-H., Math. Z., 2008) Let \((M, g)\) be a smooth compact Riemannian manifold with a conformally flat pole, \(n \geq 7\). There exists \(\delta > 0\) such that the Einstein-scalar Lichnerowicz equation

\[
\Delta_g u + \frac{n-2}{4(n-1)} S_g u = u^{2^*-1} + \frac{a}{u^{2^*+1}}
\]

is not stable on \((M, g)\) and possesses smooth positive solutions for all smooth functions \(a > 0\) such that \(\|a\|_1 < \delta\).
VI. The case \( a \geq 0 \). Unpublished result.

When \( a > 0 \) in \( M \): let \( u > 0 \) be a solution of \((EL)\). Let \( x_0 \) be such that \( u(x_0) = \min_M u \). Then \( \Delta_g u(x_0) \leq 0 \) and

\[
|h(x_0)|u(x_0) + |f(x_0)|u(x_0)^{2^* - 1} \geq \frac{a(x_0)}{u(x_0)^{2^* + 1}}
\]

\( \Rightarrow \) there exists \( \varepsilon_0 = \varepsilon_0(h, f, a) \), \( \varepsilon_0 > 0 \), such that \( u \geq \varepsilon_0 \) in \( M \).

Question: Assume \( \Delta_g + h \) is coercive, \( a \geq 0 \) and \( \max_M f > 0 \). What can we say when \( \text{Zero}(a) \neq \emptyset \) ?

In physics

\[
a = |\sigma + DW|^2 + \pi^2,
\]

where \( \sigma \) and \( \pi \) are free data, and \( W \) is the determined data given by the second equation in the system.
Recall $(EL)$ is compact if any $H^1$-bounded sequence $(u_\alpha)_\alpha$ of solutions of $(EL)$ does possess a subsequence which converges in $C^2$. Recall $(EL)$ is bounded and compact if any sequence $(u_\alpha)_\alpha$ of solutions of $(EL)$ does possess a subsequence which converges in $C^2$.

**Theorem 3:** (Druet, Esposito, H., Pacard, Pollack, Collected works - Unpublished, 2009) Assume $\Delta_g + h$ is coercive, $a \geq 0$, $a \not\equiv 0$, and $\max_M f > 0$. *Theorem 1 remains true without any other assumptions than those of Theorem 1. Assuming that $n = 3, 4, 5$, the equation is compact and even bounded and compact when $f > 0$.*

Existence follows from a combination of Theorem 1 and the sub and supersolution method. Compactness follows from the stability theorem in the proof of Theorem 2 together with an argument by Pierpaolo Esposito.
Proof of the existence part in Theorem 3: Assume the “assumptions” of Theorem 1 are satisfied: there exists \( \varphi > 0 \) such that
\[
\|\varphi\|_h^{2^*} \int_M \frac{a}{\varphi^{2^*}} \, dv_g < \frac{C(n)}{(S(h) \max_M |f|)^{n-1}} \tag{1}
\]
and \( \int_M f \varphi^{2^*} \, dv_g > 0 \). Changing \( a \) into \( a + \varepsilon_0 \) for \( 0 < \varepsilon_0 \ll 1 \), (1) is still satisfied, and since \( a + \varepsilon_0 > 0 \) we can apply Theorem 1. In particular,

(i) “\( a \to a + \varepsilon_0 \)”, \( 0 < \varepsilon_0 \ll 1 \), and Theorem 1 \( \Rightarrow \exists u_1 \) a supersolution of \((EL)\).

Now let \( \delta > 0 \) and let \( u_0 \) solve
\[
\Delta_g u_0 + hu_0 = a - \delta f^-
\]

For \( \delta > 0 \) sufficiently small, \( u_0 \) is close to the solution with \( \delta = 0 \), and since this solution is positive by the maximum principle, we get that \( u_0 > 0 \) for \( 0 < \delta \ll 1 \). Fix such a \( \delta > 0 \).
Given $\varepsilon > 0$, let $u_\varepsilon = \varepsilon u_0$. Then

$$\Delta_g u_\varepsilon + hu_\varepsilon = \varepsilon a - \delta \varepsilon f^- \leq fu_\varepsilon^{2* - 1} + \frac{a}{u_\varepsilon^{2* + 1}}$$

provided $0 < \varepsilon \ll 1$. In particular,

(ii) $u_\varepsilon = \varepsilon u_0$, $0 < \varepsilon \ll 1$, is a subsolution of (EL).

Noting that $u_\varepsilon \leq u_1$ for $\varepsilon > 0$ sufficiently small, we can apply the sub and supersolution method and get a solution $u$ to (EL) such that $u_\varepsilon \leq u \leq u_1$.

The compactness part in Theorem 3 follows from the stability theorem in the proof of Theorem 2 together with the following result by Esposito which establishes the (H2) property of Druet and Hebey under general conditions.

We do not need in what follows the $C^{1,1}_\gamma$-convergence of the $f_\alpha$’s. A $C^0$-convergence (and even less) is enough.
Lemma 4: (Esposito, Unpublished, 2009) Let \( n \leq 5 \). Let \((EL_\alpha)\) be a perturbation of \((EL)\) and \((u_\alpha)\) a sequence of solutions of \((EL_\alpha)\). Assume \( a_\alpha \geq 0 \) in \( M \) for all \( \alpha \), and \( a \neq 0 \). The \((H2)\) property holds true: \( \exists \varepsilon_0 > 0 \) such that \( u_\alpha \geq \varepsilon_0 \) in \( M \) for all \( \alpha \).

Proof of the lemma: Let \( K > 0 \) be such that \( K + h_\alpha \geq 1 \) in \( M \) for all \( \alpha \). Define \( \tilde{h}_\alpha = K + h_\alpha \) and \( \tilde{h} = K + h \). Let \( \delta > 0 \) and \( \nu_\delta, \nu^\delta \), and \( r_\alpha \) be given by

\[
\Delta_g \nu_\delta + \tilde{h}_\alpha \nu_\delta = a_\alpha - \delta f^-_\alpha ,
\]

\[
\Delta_g \nu^\delta + \tilde{h} \nu^\delta = a - \delta f^- ,
\]

\[
\Delta_g r_\alpha + \tilde{h}_\alpha r_\alpha = k_\alpha .
\]

There holds that \( \nu_\alpha^\delta \to \nu^\delta \) in \( C^0(M) \) as \( \alpha \to +\infty \) and that \( \nu^\delta \to \nu^0 \) in \( C^0(M) \) as \( \delta \to 0 \). By the maximum principle, \( \nu^0 > 0 \) in \( M \). It follows that there exists \( \delta > 0 \) sufficiently small, and \( \varepsilon_0 > 0 \), such that \( \nu_\alpha^\delta \geq \varepsilon_0 \) in \( M \) for all \( \alpha \gg 1 \). Fix such a \( \delta > 0 \). Let \( t > 0 \).
and define

\[ w_\alpha = tv_\alpha^\delta + r_\alpha. \]

We have that \( r_\alpha \to 0 \) in \( C^0(M) \) as \( \alpha \to +\infty \). There exists \( t_0 > 0 \) such that

\[
\Delta_g w_\alpha + \tilde{h}_\alpha w_\alpha = ta_\alpha - t\delta f^-_\alpha + k_\alpha
\]

\[
\leq -f^-_\alpha w^{2*-1}_\alpha + \frac{a_\alpha}{w^{2*+1}_\alpha} + k_\alpha
\]

for all \( 0 < t < t_0 \) and all \( \alpha \gg 1 \). As a consequence, since \( a_\alpha \geq 0 \) in \( M \),

\[
\Delta_g (u_\alpha - w_\alpha) + \tilde{h}_\alpha (u_\alpha - w_\alpha)
\]

\[
\geq f^-_\alpha u^{2*-1}_\alpha + f^-_\alpha w^{2*-1}_\alpha + \frac{a_\alpha}{u^{2*+1}_\alpha} - \frac{a_\alpha}{w^{2*+1}_\alpha} \geq 0
\]

for all \( \alpha \gg 1 \), at any point such that \( u_\alpha - w_\alpha \leq 0 \). The maximum principle then gives that \( w_\alpha \leq u_\alpha \) in \( M \) for all \( \alpha \gg 1 \). Since \( w_\alpha \geq \varepsilon_0 \) in \( M \) for \( \alpha \gg 1 \), this ends the proof of the lemma.