**Companion tables for the slides**

*Elliptic stability for stationary Schrödinger equations*

*I/III, II/III, III/III*,

**by E. Hebey (November 2013).**

## Unstable situations

| $(B_\alpha + R_\alpha)$-config. | $(\sum B_\alpha^i + R_\alpha)$-config. | $(u_\infty + \sum B_\alpha^i + R_\alpha)$-config. | Unbounded Energy |
|---|---|---|---|
| $(S^n, g_0)$, $n \geq 3$<br>$h_\alpha \equiv \frac{n(n-2)}{4}$<br>(Historical) | $(S^n/G, g_0)$, $n \geq 6$<br>$h_\alpha \overset{C^1}{\to} \frac{n(n-2)}{4}$<br>(Druet-Hebey, 2004) | $(S^6, g_0)$, $h > 6$<br>$h_\alpha \overset{C^1}{\to} h$, 1-Bubble,<br>$u_\infty \not\equiv 0$<br>(Druet-Hebey, 2009) | $(S^n/G, g_0)$, $n \geq 6$,<br>$h_\alpha \to \frac{n(n-2)}{4}$<br>(Druet-Hebey, 2004) |
| $(M, g)$, $n \geq 4$, $Y_g > 0$,<br>$h_\alpha \overset{C^\infty}{\to} \frac{n-2}{4(n-1)} S_g$<br>(Esposito-Pistoia<br>-Vétois, 2013) | $(M, g)$, $n \geq 6$,<br>non conf. flat, $Y_g > 0$,<br>$h_\alpha \overset{C^r}{\to} \frac{n-2}{4(n-1)} S_g$<br>(Robert-Vétois, 2013) | | $(S^n, g_0)$, $n \geq 5$, $h_\alpha \equiv \lambda$,<br>$\lambda > \frac{n(n-2)}{4}$<br>(Chen-Wei-Yan, 2011) |
| $(S^3, g_0)$, $\exists (\theta_k)_k$ res. states<br>$\theta_1 = \frac{3}{4} < \theta_k$, $k \geq 2$,<br>$\theta_k \to +\infty$, $\lambda_\alpha \to \theta_k$<br>(Hebey-Wei, 2012) | | | $(M, g)$, $n \geq 6$, non<br>conf. flat, $Y_g > 0$,<br>$h_\alpha \overset{C^r}{\to} \frac{n-2}{4(n-1)} S_g$<br>(Esposito-Pistoia<br>Vétois, Robert-Vétois,<br>2013) |
| $(S^n, \tilde{g})$, $\tilde{g}$ non conf.<br>flat, $n \geq 25$,<br>$h_\alpha \equiv \frac{n-2}{4(n-1)} S_{\tilde{g}}$<br>(Brendle, 2008;<br>Brendle-Marques, 2009) | | | |
| $(M, g)$, $n \geq 4$, conf. flat,<br>$h_\alpha \overset{C^\infty}{\to} \frac{n-2}{4(n-1)} \max_M S_{\tilde{g}}$<br>for some $\tilde{g} \in [g]$, $S_{\tilde{g}}$ max.<br>at only one point<br>(Hebey-Vaugon, 2001) | | | |

## Stable situations

| Bounded stability | Analytic stability |
|---|---|
| $(M, g)$, $n \geq 3$, $Y_g > 0$,<br>$h < \frac{n-2}{4(n-1)} S_g$<br>(Li-Zhu, $n = 3$, 1999; Druet,<br>$n \geq 4$, 2004; see also<br>Druet-Hebey-Vétois, 2010) | $(M, g)$, $n \geq 4$<br>$\Delta_g + h$ coercive, $n \neq 6$,<br>$h \neq \frac{n-2}{4(n-1)} S_g$<br>(Druet, 2003; see also<br>Druet-Hebey, 2009) |
| Compactness | |
| $(M, g)$ conf. flat or $3 \leq n \leq 24$,<br>$Y_g > 0$, $(M, g) \neq (S^n, g_0)$,<br>$h_\alpha \equiv \frac{n-2}{4(n-1)} S_g$<br>(Schoen, 1991; Khuri-Marques-<br>Schoen, 2009) | |

## A general reference


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