Elliptic stability for stationary Schrödinger equations

by

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Part V/VI

Proofs

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Nonlinear analysis arising from geometry and physics

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PART V. PROOFS OF THE BOUNDED STABILITY AND ANALYTIC STABILITY THEOREMS.

II.1) Proof of the bounded stability theorem.

II.2) Proof of the analytic stability theorem.

NOTE : The blue writing is what you have to write down to be able to follow the slides presentation.

PART V. PROOFS.

V.1) Proof of the bounded stability theorem :

We discuss the proof of the bounded stability theorem. The proof follows the scheme of the original proofs by Schoen and Li-Zhu (1999). We follow here the presentation by Druet (2004). A recent very nice reorganisation of the proof is by Druet and Premoselli (2014). The equation is

$$\Delta_g u + hu = u^{2^\star - 1} .$$

We require that $\Delta_g + h$ is coercive (it has to be nonnegative if we want the u_α's to exist), and we require that for any $x \in M$,

$$h(x) < \frac{n-2}{4(n-1)} S_g(x) . \qquad (H)$$

The model case then is a sequence

$$\Delta_g u + h_\alpha u = u^{2^\star - 1}$$

of critical stationary Schrödinger equations, with $h_\alpha \to h$ in C^1, and a sequence $(u_\alpha)_\alpha$ of solutions of these equations with no energy assumptions at all. We assume by contradiction that $\|u_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$. Here $n \geq 3$.

We look for blow-up points like the ones which would be given by the H^1-theory if the u_α's were bounded in H^1. A very general result is as follows : given $u \in C^1(M, \mathbb{R})$, there exist $N \in \mathbb{N}^\star$ and a family $(x_1, \dots, x_N)$ of critical points of u such that

$$d_g(x_i, x_j)^{\frac{n-2}{2}} u(x_i) \geq 1$$

for all $i \neq j$, and

$$\left(\min_i d_g(x_i, x)\right)^{\frac{n-2}{2}} u(x) \leq 1$$

for all critical points of u. From this result we can prove that for any α, there exist $N_\alpha \in \mathbb{N}^\star$, and critical points $x_{1,\alpha}, \dots, x_{N_\alpha,\alpha}$ of u_α such that

$$d_g(x_{i,\alpha}, x_{j,\alpha})^{\frac{n-2}{2}} u_\alpha(x_{i,\alpha}) \geq 1$$

for all $i \neq j$, and

$$\left(\min_i d_g(x_{i,\alpha}, x)\right)^{\frac{n-2}{2}} u_\alpha(x) \leq C$$

for some $C > 0$ and not only all critical points of u_α, but for all $x \in M$. The first goal then is to prove that the following lemma holds true.

Lemma : Assume nothing when $n = 3$ or (H) when $n \geq 4$. Then blow-up points are isolated.

Proof : We define

$$d_\alpha = \min_{1 \leq i < j \leq N_\alpha} d_g(x_{i,\alpha}; x_{j,\alpha}) \ ,$$

and if $N_\alpha = 1$, we set $d_\alpha = \frac{i_g}{4}$, where i_g is the injectivity radius. We want to prove that $d_\alpha \not\to 0$ as $\alpha \to +\infty$. We proceed by contradiction and we assume that

$$d_\alpha \to 0$$

as $\alpha \to +\infty$. Then $N_\alpha \geq 2$, and we order $x_{1,\alpha}, \dots, x_{N_\alpha,\alpha}$ such that

$$d_\alpha = d_g(x_{1,\alpha}, x_{2,\alpha}) \leq d_g(x_{1,\alpha}, x_{3,\alpha}) \leq \dots \leq d_g(x_{1,\alpha}, x_{N_\alpha,\alpha}) \ .$$

Given $R > 0$ we define $N_{R,\alpha} \in \{1, \dots, N_\alpha\}$ to be such that

$$\begin{aligned} d_g(x_{1,\alpha}, x_{i,\alpha}) &\leq R d_\alpha \quad \text{for all } 1 \leq i \leq N_{R,\alpha} \\ d_g(x_{1,\alpha}, x_{i,\alpha}) &> R d_\alpha \quad \text{for all } i > N_{R,\alpha} \ . \end{aligned}$$

Obviously, $N_{R,\alpha} \geq 2$ when $R > 1$. Also we can prove that $(N_{R,\alpha})_\alpha$ is bounded for all $R > 1$ since $B_{x_{i,\alpha}}(d_\alpha/2) \bigcap B_{x_{j,\alpha}}(d_\alpha/2) = \emptyset$ for all $i \neq j$, and then $\mathrm{Vol}_g\left(B_{x_{1,\alpha}}(\frac{3R}{2} d_\alpha)\right) \geq \sum_{i=1}^{N_{R,\alpha}} \mathrm{Vol}_g\left(B_{x_{i,\alpha}}(d_\alpha/2)\right)$ so that we get an upper bound on $N_{R,\alpha}$ depending only on R.

Also we can prove that for any $R > 1$, and any $i = 1, \ldots, N_{R,\alpha}$,

$$d_\alpha^{\frac{n-2}{2}} u_\alpha(x_{i,\alpha}) \to +\infty$$

as $\alpha \to +\infty$. Then for $x_\alpha = x_{1,\alpha}$ and $\rho_\alpha = \frac{d_\alpha}{8}$ there holds that

$$\begin{cases} \nabla u_\alpha(x_\alpha) = 0 \ \text{ for all } \alpha\ , \\ d_g(x_\alpha, x)^{\frac{n-2}{2}} u_\alpha(x) \leq C \ \text{ for all } \alpha \text{ and all } x \in B_{x_\alpha}(7\rho_\alpha)\ , \\ \lim_{\alpha\to+\infty} \rho_\alpha^{\frac{n-2}{2}} \sup_{B_{x_\alpha}(6\rho_\alpha)} u_\alpha = +\infty\ , \end{cases} \qquad (BS)$$

for some $C > 0$ independent of α and x. There holds that $u_\alpha(x_\alpha) \to +\infty$ as $\alpha \to +\infty$. Let $\mu_\alpha = u_\alpha(x_\alpha)^{-\frac{2}{n-2}}$ and define $\mu_\alpha \ll r_\alpha \leq \rho_\alpha$ be the radius up to which the bubble $(B_\alpha)_\alpha$ of center x_α and weight μ_α does not see any other kind of bubble (decreasing condition). We can prove a local C^0-theory around x_α up to r_α with one bubble $(B_\alpha)_\alpha$. Essentially we can prove that

$$u_\alpha(x) + d_g(x_\alpha, x)|\nabla u_\alpha(x)| \leq C\mu_\alpha^{\frac{n-2}{2}} d_g(x_\alpha, x)^{2-n}$$

for all α and all $x \in B_{x_\alpha}(r_\alpha)\backslash\{x_\alpha\}$, and that

$$|u_\alpha(x) - B_\alpha(x)| \leq C\mu_\alpha^{\frac{n-2}{2}} \left(r_\alpha^{2-n} + d_g(x_\alpha, x)^{3-n}\right) + \varepsilon_\alpha B_\alpha(x)$$

for all α and all $x \in B_{x_\alpha}(r_\alpha)\backslash\{x_\alpha\}$. It follows from these estimates that :

if $r_\alpha \to 0$, then

$$r_\alpha^{n-2}\mu_\alpha^{1-\frac{n}{2}}u_\alpha\left(\exp_{x_\alpha}(r_\alpha x)\right) \to \frac{(n(n-2))^{\frac{n-2}{2}}}{|x|^{n-2}} + \mathcal{H}(x)$$

in $C^2_{loc}(B_0(2)\backslash\{0\})$, where $\mathcal{H}$ is harmonic in $B_0(2)$. Then we use a Pohozaev identity in $B_{x_\alpha}(r_\alpha)$ and we get that

(i) if $n = 3$ and $r_\alpha \to 0$, then $\rho_\alpha = O(r_\alpha)$ and $\mathcal{H}(0) = 0$

(ii) when $n \geq 4$ and we assume (H), it is always the case that $r_\alpha \to 0$, and we have that $\rho_\alpha = O(r_\alpha)$ and $\mathcal{H}(0) \leq 0$.

Now we return to the configuration $x_\alpha = x_{1,\alpha}$ and $\rho_\alpha = \frac{d_\alpha}{8}$. We let $\hat{u}_\alpha$ be given by

$$\hat{u}_\alpha(x) = d_\alpha^{\frac{n-2}{2}}u_\alpha\left(\exp_{x_{1,\alpha}}(d_\alpha x)\right)$$

and let $v_\alpha = d_\alpha^{\frac{n-2}{2}}\mu_\alpha^{1-\frac{n}{2}}\hat{u}_\alpha$. By the above,

$$v_\alpha(x) \to \frac{\Lambda}{|x|^{n-2}} + \mathcal{H}(x)$$

around 0, where $\Lambda > 0$ and $\mathcal{H}(0) \leq 0$. On the other hand the v_α's satisfy an equation like

$$\Delta_{g_\alpha} v_\alpha + d_\alpha^2 \hat{h}_\alpha v_\alpha = \varepsilon_\alpha v_\alpha^{2^\star - 1}$$

where $g_\alpha \to \delta$, $\hat{h}_\alpha(x) = h_\alpha\left(\exp_{x_{1,\alpha}}(d_\alpha x)\right)$, and $\varepsilon_\alpha \to 0$. Let

$$\hat{x}_{i,\alpha} = \frac{1}{d_\alpha} \exp_{x_{1,\alpha}}^{-1}(x_{i,\alpha})$$

for $i = 1, \dots, N_{R,\alpha}$. Since $(N_{R,\alpha})_\alpha$ is bounded for any $R > 0$, we get a locally finite at most countable collection $\{\hat{x}_i\}_{i \in I}$ of points in $\mathbb{R}^n$ such that

$$v_\alpha \to G$$

in $C^1_{\text{loc}}(\mathbb{R}^n \backslash \{\hat{x}_i\}_{i \in I})$, where, by the equation satisfied by the v_α's, $\Delta G = 0$ in $\mathbb{R}^n \backslash \{\hat{x}_i\}_{i \in I}$. Hence

$$G(x) = \sum_{i=1}^{N_R} \frac{\Lambda_i}{|x - \hat{x}_i|^{n-2}} + H_R(x)$$

in $B_0(R)$, where $\Lambda_i > 0$ and H_R is harmonic in $B_0(R)$. Then, around 0,

$$G(x) = \frac{\Lambda_1}{|x|^{n-2}} + X(x) ,$$

where $X(x) = \sum_{i=2}^{N_R} \frac{\Lambda_i}{|x - \hat{x}_i|^{n-2}} + H_R(x)$. In particular,

$$X(x) = \frac{\Lambda_2}{|x - \hat{x}_2|^{n-2}} + Y(x) \ .$$

There holds that $|\hat{x}_2| = 1$ (since $d_g(x_{1,\alpha}, x_{2,\alpha}) = d_\alpha$). We also have that $G \geq 0$ and Y is harmonic in $B_0(R)\backslash\{\hat{x}_3, \ldots, \hat{x}_{N_R}\}$. By the maximum principle,

$$Y(0) \geq \min_{|x|=R} Y(x)$$

and since

$$\frac{\Lambda_1}{|x|^{n-2}} + \frac{\Lambda_2}{|x - \hat{x}_2|^{n-2}} + Y(x) = G(x)$$

and $G \geq 0$, we get that

$$Y(0) \geq -\frac{\Lambda_1}{R^{n-2}} - \frac{\Lambda_2}{(R-1)^{n-2}} \ .$$

Hence, around 0,

$$v_\alpha(x) \to \frac{\Lambda_1}{|x|^{n-2}} + X(x)$$

in C^1_{loc}, and

$$X(0) \geq \Lambda_2 - \frac{\Lambda_1}{R^{n-2}} - \frac{\Lambda_2}{(R-1)^{n-2}} \ .$$

Picking $R \gg 1$, we get that $X(0) \geq 0$. Summarizing $v_\alpha \to v$ and, around 0,

$$\begin{cases} v(x) = \frac{\Lambda}{|x|^{n-2}} + \mathcal{H}(x) \ , \\ v(x) = \frac{\Lambda_1}{|x|^{n-2}} + X(x) \ , \end{cases}$$

where $\Lambda, \Lambda_1 > 0$, $\mathcal{H}(0) \leq 0$ and $X(0) \geq 0$. A contradiction, and this ends the proof of the Lemma stating that blow-up points are isolated. ■

Now we can conclude to the proof of the bounded stability theorem. The conclusion uses the positive mass theorem when $n = 3$ and is slightly easier when $n \geq 4$.

Proof of the theorem when $n \geq 4$ **:** We let $(u_\alpha)_\alpha$ be such that

$$\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{2^\star - 1}$$

in M for all α. We assume that $h_\alpha \to h$ in C^1 and $h < \frac{n-2}{4(n-1)} S_g$. By contradiction we assume that $\|u_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$. Let x_α be a point where u_α is maximum. Up to passing to a subsequence, since $d_\alpha \not\to 0$, there exists i such that

$$d_g(x_{i,\alpha}, x_\alpha) \to 0$$

as $\alpha \to +\infty$. Moreover, $d_g(x_{i,\alpha}, x)^{\frac{n-2}{2}} u_\alpha(x) \leq C$ for all $x \in B_{x_i}(\delta)$, where $x_{i,\alpha} \to x_i$ and $0 < \delta \ll 1$. Then, by the definition of x_α,

$$
\begin{aligned}
& d_g(x_\alpha, x) u_\alpha(x)^{\frac{2}{n-2}} \\
& \leq d_g(x_{i,\alpha}, x) u_\alpha(x)^{\frac{2}{n-2}} + d_g(x_{i,\alpha}, x_\alpha) u_\alpha(x)^{\frac{2}{n-2}} \\
& \leq C + d_g(x_{i,\alpha}, x_\alpha) u_\alpha(x_\alpha)^{\frac{2}{n-2}} \\
& \leq 2C
\end{aligned}
$$

for all $x \in B_{x_i}(\delta)$. In particular,

$$
\begin{cases}
\nabla u_\alpha(x_\alpha) = 0 \ \text{ for all } \alpha \ , \\
d_g(x_\alpha, x)^{\frac{n-2}{2}} u_\alpha(x) \leq C \ \text{ for all } \alpha \text{ and all } x \in B_{x_\alpha}(7\rho_\alpha) \ , \\
\lim_{\alpha \to +\infty} \rho_\alpha^{\frac{n-2}{2}} \sup_{B_{x_\alpha}(6\rho_\alpha)} u_\alpha = +\infty \ ,
\end{cases}
\qquad (BS)
$$

holds true with x_α and $\rho_\alpha = \delta$ for $0 < \delta \ll 1$ fixed. However we have seen that $\rho_\alpha = O(r_\alpha)$ and $r_\alpha \to 0$. A contradiction. This proves the bounded stability theorem when $n \geq 4$. ■

V.2) Proof of the analytic stability theorem :

We discuss the proof of the analytic stability theorem. The equation is

$$\Delta_g u + hu = u^{2^\star - 1} \ .$$

We require that $\Delta_g + h$ is coercive (it has to be nonnegative if we want the u_α's to exist), and we require that for any $x \in M$,

$$h(x) \neq \frac{n-2}{4(n-1)} S_g(x) \ . \qquad (H)$$

These two conditions make that we can deal with large h's. The model case then is a sequence

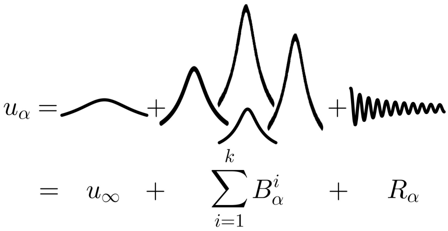
$$\Delta_g u + h_\alpha u = u^{2^\star - 1}$$

of critical stationary Schrödinger equations, with $h_\alpha \to h$ in C^1, and a sequence $(u_\alpha)_\alpha$ of solutions of these equations with $\|u_\alpha\|_{H^1} = O(1)$. We assume by contradiction that $\|u_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$. Here $n \geq 4$.

By the H^1-theory,

$$u_\alpha = u_\infty + \sum_{i=1}^{k} B_\alpha^i + R_\alpha \ ,$$

where u_∞, the limit profile function solves the limit system, the $(B_\alpha^i)_\alpha$'s are bubbles, and $R_\alpha \to 0$ in H^1 as $\alpha \to +\infty$. A priori we can have accumulations of bubbles (contrary to the bounded stability case where such a situation is impossible). On a drawing the H^1-theory tells us that

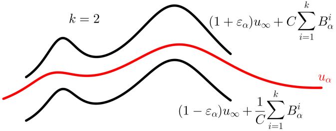


The H^1-theory is not enough to prove the analytic stability theorem, we need the C^0-theory.

By the C^0-theory, there exist $C > 1$ and a sequence $(\varepsilon_\alpha)_\alpha$ converging to zero such that

$$(1-\varepsilon_\alpha)u_\infty(x) + \frac{1}{C}\sum_{i=1}^{k} B_\alpha^i(x) \leq u_\alpha(x) \leq (1+\varepsilon_\alpha)u_\infty(x) + C\sum_{i=1}^{k} B_\alpha^i(x)$$

for all $x \in M$ and all α. In other words, for instance when $k = 2$,

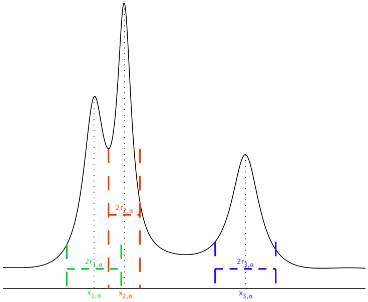


From this control we get sharp exact asymptotics for the u_α's :

$$u_\alpha = \big(1 + o(1)\big)\, u_\infty + \sum_{i=1}^{k} \big(\Phi(x_i, \cdot) + o(1)\big)\, B_\alpha^i \ ,$$

where $\Phi \in C^0\left(M \times M, \mathbb{R}\right)$ equals one on the diagonal.

The range of interaction $r_{i,\alpha}$ of a bubble $(B^i_\alpha)_\alpha$ is the radius up to which the bubble $(B^i_\alpha)_\alpha$ is leading and at which it starts interacting with another bubble. Assume for instance that we have 3 bubbles (2 interacting + 1), that $u_\infty \not\equiv 0$, that $d_g(x_{1,\alpha}, x_{2,\alpha}) = o(\mu_{1,\alpha})$ and that $\mu_{2,\alpha} = o(\mu_{1,\alpha})$. Then we get that



Namely $r_{1,\alpha} = \sqrt{\mu_{1,\alpha}}$, $r_{3,\alpha} = \sqrt{\mu_{3,\alpha}}$, and $r_{2,\alpha} \sim \sqrt{\mu_{1,\alpha}\mu_{2,\alpha}}$. As we can check, $B^1_\alpha(x_\alpha) = B^2_\alpha(x_\alpha)$ if and only if $d_g(x_{2,\alpha}, x_\alpha) \sim r_{2,\alpha}$.

Let μ_α be the maximum weight given by $\mu_\alpha = \max_i \mu_{i,\alpha}$. By the C^0-theory, if $r_{i,\alpha} = o\left(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\right)$, then

$$r_{i,\alpha}^{n-2}\mu_{i,\alpha}^{1-\frac{n}{2}} u_\alpha\left(\exp_{x_{i,\alpha}}(r_{i,\alpha}x)\right) \to \frac{(n(n-2))^{\frac{n-2}{2}}}{|x|^{n-2}} + \mathcal{H}_i(x)$$

in $C^2_{loc}\left(B_0(\delta)\backslash\{0\}\right)$ as $\alpha \to +\infty$ for $0 < \delta \ll 1$, where $\mathcal{H}_i$ is a harmonic function in $B_0(\delta)$ satisfying that $\mathcal{H}_i(0) \neq 0$ and $\nabla\mathcal{H}_i(0) \equiv 0$. Moreover, when $\mu_{i,\alpha} \sim \max_j \mu_{j,\alpha}$, then

$$\mathcal{H}_i(x) = \sum_{j\in I_i} \frac{\lambda_{i,j}}{|x - x_{i,j}|^{n-2}} + \left(\lim_{\alpha\to+\infty} r_{i,\alpha}^{n-2}\mu_{i,\alpha}^{1-\frac{n}{2}}\right) u_\infty(x_i) ,$$

where x_i is the limit of the $x_{i,\alpha}$'s,$\lambda_{i,j} > 0$ for all i, j, the $x_{i,j}$'s are given by

$$x_{i,j} = \lim_{\alpha\to+\infty} \frac{1}{r_{i,\alpha}} \exp^{-1}_{x_{i,\alpha}}(x_{j,\alpha}) ,$$

and (to make things simples) I_i consist of the j's such that $\mu_{j,\alpha} \sim \mu_{i,\alpha}$. When $\mu_{i,\alpha} \sim \max_j \mu_{j,\alpha}$, the condition $r_{i,\alpha} = o\left(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\right)$ means nothing but that $r_{i,\alpha} \to 0$.

The Pohozaev identity around the $x_{i,\alpha}$'s at a scale like $r_{i,\alpha} \Rightarrow$ when $n = 4$ that

$$\left(h(x_i) - \frac{1}{6} S_g(x_i) + o(1)\right) \mu_{i,\alpha}^2 \ln \frac{r_{i,\alpha}}{\mu_{i,\alpha}}$$
$$= O\left(\mu_{i,\alpha}\mu_\alpha\right) + o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) \quad \text{in general} ,$$
$$= C_0 \left(\mathcal{H}_i(0) + o(1)\right) \mu_{i,\alpha}^2 r_{i,\alpha}^{-2} + o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right)$$

when $r_{i,\alpha} = o\left(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\right)$, where $C_0 > 0$ is dimensional constant, and $h = \lim h_\alpha$'s. And when $n \geq 5$,

$$\left(h(x_i) - \frac{n-2}{4(n-1)} S_g(x_i) + o(1)\right) \mu_{i,\alpha}^2$$
$$= O\left(\mu_{i,\alpha}^{\frac{n-2}{2}} \mu_\alpha^{\frac{n-2}{2}}\right) \quad \text{in general} ,$$
$$= C_0 \left(\mathcal{H}_i(0) + o(1)\right) \mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n}$$

when $r_{i,\alpha} = o\left(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\right)$, where $C_0 > 0$, h are as above.

We also have by the very first definition of the range of influence that

$$r_{i,\alpha} \leq \sqrt{\mu_{i,\alpha}} \ \text{ if } \ u_\infty \not\equiv 0 \ .$$

In particular, $r_{i,\alpha} \to 0$ as $\alpha \to +\infty$ if $u_\infty \not\equiv 0$. Picking i such that $\mu_{i,\alpha} \sim \max_j \mu_{j,\alpha}$, the condition $r_{i,\alpha} = o\left(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\right)$ is satisfied. Then it follows from the Pohozaev expansions that

$$\left(\mathcal{H}_i(0) + o(1)\right)\mu_{i,\alpha}^2 r_{i,\alpha}^{-2} = O\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) \text{ if } n = 4 \ ,$$
$$\left(\mathcal{H}_i(0) + o(1)\right)\mu_{i,\alpha}^{n-2} r_{i,\alpha}^{2-n} = O\left(\mu_{i,\alpha}^2\right) \text{ if } n \geq 5 \ .$$

Since $\mu_{i,\alpha}^2 r_{i,\alpha}^{-2} \geq C\mu_{i,\alpha}$, we get that $\mathcal{H}_i(0) = 0$ when $n = 4$ and $n = 5$, a contradiction with $\mathcal{H}_1(0) \neq 0$. This proves that

$$u_\infty \equiv 0$$

when $n = 4, 5$.

Still by the Pohozaev expansions, it follows from (H) that $r_{i,\alpha} \to 0$ for all i such that $\mu_{i,\alpha} \sim \max_j \mu_{j,\alpha}$. Let I_1 be the set of such indices i, $\mu_{1,\alpha} = \max_j \mu_{j,\alpha}$, and let $i \in I_1$ be such that

$$d_g(x_{1,\alpha}, x_{i,\alpha}) \geq d_g(x_{1,\alpha}, x_{j,\alpha})$$

for all $j \in I_1$. Then the $x_{i,j}$'s all lie in a ball in the Euclidean space whose boundary contains 0, and they are not 0. In particular, for this i, there exists a vector $\nu_i \in \mathbb{R}^n$ such that $|\nu_i| = 1$ and $\langle x_{i,j}, \nu_i \rangle > 0$ for all $j \in I_1$. By the expression for $\mathcal{H}_i$, namely

$$\mathcal{H}_i(x) = \sum_{j \in I_i} \frac{\lambda_{i,j}}{|x - x_{i,j}|^{n-2}} + \left(\lim_{\alpha \to +\infty} r_{i,\alpha}^{n-2} \mu_{i,\alpha}^{1-\frac{n}{2}} \right) u_\infty(x_i) \ ,$$

we get that

$$\nabla \mathcal{H}_i(0).\nu_i = \sum_j \frac{\lambda_{i,j} \langle x_{i,j}, \nu_i \rangle}{|x_{i,j}|^n} \ ,$$

a contradiction with $\nabla \mathcal{H}_i(0) \equiv 0$. This proves the analytic stability theorem. ■

Thank you for your attention !