Elliptic stability for stationary Schrödinger equations
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Part II/VI
An introduction to elliptic stability
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Nonlinear analysis arising from geometry and physics
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PART II. AN INTRODUCTION TO ELLIPTIC STABILITY.

II.1) The model equation.

II.2) Equations behind the model equation.

II.3) A first insight into elliptic stability.

II.4) The subcritical world.

II.5) More precise definitions are needed in the critical world.
NOTE: The blue writing is what you have to write down to be able to follow the slides presentation.
(M, g) smooth compact, ∂M = ∅ (closed manifold), n ≥ 3.

Model equation
\[ \Delta_g u + hu = u^{p-1} \quad (E_h) \]
Varying h's

Here : u ≥ 0, \( \Delta_g = -\text{div}_g \nabla \), h ∈ \( C^{0,\theta} \) (typically), \( p \in (2, 2^\ast] \), where \( 2^\ast = \frac{2n}{n-2} \) is the critical Sobolev exponent. \( H^1 \) Sobolev space of functions in \( L^2 \) with one derivative in \( L^2 \). Then \( H^1 \subset L^p \) for all \( p \leq 2^\ast \), and

\[ H^1 \subset L^p \text{ is compact when } p < 2^\ast, \]
but not when \( p = 2^\ast \).

Subcritical “world” \( p < 2^\ast \) \neq \text{ Critical “world” } \ p = 2^\ast

Question : How much is \( (E_h) \) robust with respect to \( h \)?
II.2) Equations behind the model equation:

- The Yamabe equation
- The stationary Klein-Gordon-Maxwell-Proca system
- The Einstein-Lichnerowicz equation
- The Kirchhoff equation

The Yamabe equation is obviously of the $(E_h)$-type. It is written as

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = u^{2^*-1}.$$  \hfill (Y)

We get an equation like $(E_h)$, where

$$h = \frac{n-2}{4(n-1)} S_g$$

is given by the geometry of the manifold (and $p = 2^*$ is critical). As we saw, the LHS in $(Y)$ is the conformal Laplacian (and it enjoys conformal invariance).
The stationary Klein-Gordon-Maxwell-Proca system in reduced form is also of the \((E_h)\)-type. The \((\text{KGMP}_r)\)-system is written as

\[
\begin{aligned}
\Delta_g u + m_0^2 u &= u^{2*-1} + \omega^2 (qv - 1)^2 u \\
\Delta_g \nu + (m_1^2 + q^2 u^2) \nu &= qu^2.
\end{aligned}
\]

We can always solve the equation

\[
\Delta_g \Phi(u) + (m_1^2 + q^2 u^2) \Phi(u) = qu^2
\]

and we get a map \(\Phi : H^1 \rightarrow H^1\). Then the \((\text{KGMP}_r)\)-system reduces to the first equation

\[
\Delta_g u + m_0^2 u = u^{2*-1} + \omega^2 (q\Phi(u) - 1)^2 u.
\]

The solutions of the system are the couples \((u, \Phi(u))\). We get an equation like \((E_h)\), where \(h\) is now given by

\[
h = m_0^2 - \omega^2 (q\Phi(u) - 1)^2.
\]

In particular, \(h\) depends on \(u\), and (in the 3d-model) \(h\) turns out to be controlled in \(C^{0,\theta}\)-topologies.

The Einstein-scalar field Lichnerowicz equation corresponds to the Hamiltonian constraint in the constraint equations in the conformal method setting (Lichnerowicz). The two constraint equations
(Hamiltonian + Momentum) are written (conformal method setting) as:

\[
\begin{align*}
\Delta_g u + h_0 u &= f u^{2^*-1} + \frac{a}{u^{2^*+1}} \quad \text{(EL)} \\
\overrightarrow{\Delta}_g X &= \frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi \quad \text{(MC)}
\end{align*}
\]

where \( h_0, f \) and \( a \) are given (depending on the geometry and physics data), \( u \) is an unknown function, \( X \) is an unknown vector field, and \( \overrightarrow{\Delta}_g = \nabla . \mathcal{L} \) (\( \mathcal{L} \) the conformal Killing operator). The (EL)-equation is the Einstein-Lichnerowicz equation. It is highly nonlinear and, in the CMC-case (where \( \tau = C^{st} \)) it fully describes the (CE)-system, since then the two equations are independent (and (MC) is a “basic” Laplace type equation).

The negative power term in (EL) \( \Rightarrow \) there exists \( \varepsilon_0 > 0 \) s.t. \( u \geq \varepsilon_0 \) for all solution of the Hamiltonian constraint. A very basic argument when \( a > 0 \) is as follows: let \( x_0 \) be a point where \( u \) is minimum. Then \( \Delta_g u(x_0) \leq 0 \) and we get from (EL) that

\[
\frac{a(x_0)}{u(x_0)^{2^*+1}} + f(x_0) u(x_0)^{2^*-1} \leq h_0(x_0) u(x_0)
\]

and when \( a > 0 \) this obviously implies that there exists \( \varepsilon_0 > 0 \), independent of \( u \), such that \( u \geq \varepsilon_0 \) in \( \mathcal{M} \).
In specific cases $f \equiv 1$. Then we recover an equation like $(E_h)$, where

$$h = h_0 - \frac{a}{u^{2*+2}}.$$

In particular $h$ depends again on $u$, and $h$ is here controlled in the $L^\infty$-topology.

The Kirchhoff equation is written as

$$\left(a + b \int_M |\nabla u|^2 dv_g\right) \Delta_g u + h_0 u = u^{2*-1}, \quad (K)$$

where $a, b > 0$ are positive real numbers and $h_0 \in C^1(M, \mathbb{R})$. Let $K(u) = a + b \int_M |\nabla u|^2 dv_g$, and define $\nu = K(u)^{-\frac{1}{2*+2}} u$. Then

$$\Delta_g \nu + h \nu = \nu^{2*-1},$$

and we recover an equation like $(E_h)$, where $h = \frac{h_0}{K(u)}$. Here again $h$ depends on $u$, and $h$ is in this case controlled in the $C^1$-topology.

Moral: There are several models hidden in our model equation $(E_h)$ when $h$ depends on the solution $u$. The sole control on the set in which $h$ varies will have to matter in our approach.
II.3) A first insight into elliptic stability:

Consider equations like

$$\Delta_g u = f(x, u), \quad (E)$$

where $f : M \times \mathbb{R} \to \mathbb{R}$ is given, and the Laplacian $\Delta_g = -\text{div}_g \nabla$ is the Laplace-Beltrami operator.

Goal: define the stability (robustness) of $(E)$ with respect to $f$.

Let $S_f$ be the set of solutions of $(E)$. Let $\mathcal{P}$ be a set of perturbations of $f$, namely a family of functions $\tilde{f} : M \times \mathbb{R} \to \mathbb{R}$ such that $\tilde{f} \in \mathcal{P}$. For the sake of simplicity we assume $S_{\tilde{f}} \subset C^2$ for all $\tilde{f} \in \mathcal{P}$. Define the pointed distance between subsets of $C^2$ by

$$d_{C^2}^\rightarrow(X; Y) = \sup_{v \in X} \inf_{u \in Y} \|v - u\|_{C^2},$$

and we adopt the conventions that $d_{C^2}^\rightarrow(X; \emptyset) = +\infty$ if $X \neq \emptyset$, and $d_{C^2}^\rightarrow(\emptyset; Y) = 0$ for all $Y$. Then, $d_{C^2}^\rightarrow(X; Y) = 0$ iff $X \subset \overline{Y}$, and $d_{C^2}^\rightarrow$ satisfies the triangle inequality

$$d_{C^2}^\rightarrow(X; Z) \leq d_{C^2}^\rightarrow(X; Y) + d_{C^2}^\rightarrow(Y; Z)$$

for all $X, Y, Z \subset C^2$. 
We consider
\[ \Delta_g u = f(x, u) , \quad (E) \]
and define two notions of stability for (E).

**Definition : (Geometric and Analytic stability)**

Equation (E) is geometrically stable with respect to a set \( \mathcal{P} \) of perturbations of \( f \) and a norm \( \| \cdot \|_\mathcal{P} \) on \( \mathcal{P} \) if

\[
\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall \tilde{f} \in \mathcal{P}, \| \tilde{f} - f \|_\mathcal{P} < \delta \Rightarrow d_{C^2}(S_{\tilde{f}}; S_f) < \varepsilon ;
\]

Equation (E) is analytically stable with respect to \( \mathcal{P} \) and \( \| \cdot \|_\mathcal{P} \) if for any sequence \( (f_\alpha)_\alpha \) in \( \mathcal{P} \), converging to \( f \) w.r.t. \( \| \cdot \|_\mathcal{P} \) as \( \alpha \to +\infty \), and any sequence \( (u_\alpha)_\alpha \) of solutions of \( \Delta_g u_\alpha = f_\alpha(\cdot, u_\alpha) \) in \( M \), there holds that, up to a subsequence, \( u_\alpha \to u \) in \( C^2 \) as \( \alpha \to +\infty \), where \( u \) solves (E).

Geometric stability expresses the fact that \( S_f \) is stable with respect to perturbations of \( f \). It corresponds to the continuity in \( \mathcal{P} \) of the function \( \tilde{f} \to d_{C^2}(S_{\tilde{f}}; S_f) \). It is easily checked (by contradiction) that :

\[
\text{Analytic stability } \Rightarrow \text{ Geometric stability} .
\]

The converse is false in general as we can prove below.
An example of a geometrically stable equation which turns out to be not analytically stable: Let $\lambda_1 \in \text{Sp}(\Delta_g)$ be the first nonzero eigenvalue of $\Delta_g$, $\lambda_1 > 0$. Let $u_0 \not\equiv 0$ and $f_0 \not\equiv 0$ be smooth functions satisfying that $\Delta_g u_0 - \lambda_1 u_0 = f_0$, and consider the equation

$$\Delta_g u - \lambda_1 u = f_0 \quad (E')$$

Then $u_0$ solves $(E')$. We let $P = \{\tilde{f}(\cdot, u) = f(\cdot) + \lambda u, \lambda \in \mathbb{R}, f \in C^{0,\theta}\}$, and define $\| \cdot \|_P$ by

$$\| \tilde{f} \|_P = |\lambda| + \| f \|_{C^{0,\theta}}.$$ 

In other words, we perturb $(E')$ by perturbing $\lambda_1$ and $f_0$ in $\mathbb{R} \times C^{0,\theta}$.

**Claim 1**: $(E')$ is not analytically stable (and not even compact). We see this by picking $\varphi \not\equiv 0$ in the eigenspace associated to $\lambda_1$. We let $(k_\alpha)_{\alpha}$ be a sequence of positive real numbers s.t. $k_\alpha \to +\infty$ as $\alpha \to +\infty$. We define

$$u_\alpha = u_0 + k_\alpha \varphi.$$ 

Obviously, the $u_\alpha$'s all solve $(E')$. However $\| u_\alpha \|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$, and this contradicts the analytic stability of $(E')$. 

Elliptic stability - Part II - An introduction to Elliptic Stability
Claim 2: We claim that \((E')\) is geometrically stable (w.r.t. perturbations of \(\lambda_1\) and \(f_0\) in \(\mathbb{R} \times C^{0,\theta}\)). We prove this by contradiction. Then there exists \(\varepsilon_0 > 0\), a sequence \((\lambda_\alpha)\) such that \(\lambda_\alpha \to \lambda_1\) as \(\alpha \to +\infty\), and a sequence \((f_\alpha)\) such that \(f_\alpha \to f_0\) in \(C^{0,\theta}\) as \(\alpha \to +\infty\), with the property that
\[
d_{C^2}(S(\lambda_\alpha, f_\alpha); S(\lambda_1, f_0)) \geq \varepsilon_0 ,
\]
where \(S(\lambda, f)\) stands for the set of solutions of \(\Delta_g u - \lambda u = f\) (so that \(S(\lambda_1, f_0)\) is precisely the set of solutions of \((E')\)). In particular, it follows from \((\ast)\) that there exists a sequence \((u_\alpha)\) of \(C^2\)-functions such that
\[
\Delta_g u_\alpha - \lambda_\alpha u_\alpha = f_\alpha \quad (E_\alpha)
\]
for all \(\alpha\), and such that \(d_{C^2}(u_\alpha; S(\lambda_1, f_0)) \geq \frac{\varepsilon_0}{2}\) for all \(\alpha\). Let \(E_{\lambda_1}\) be the eigenspace of \(\Delta_g\) associated to \(\lambda_1\). We know \(E_{\lambda_1}\) is finite dimensional. We let \(\varphi_1, \ldots, \varphi_k\) be a \(L^2\)-orthonormal basis for \(E_{\lambda_1}\), and let \(v_\alpha\) and \(\varphi_\alpha\) be given by
\[
v_\alpha = u_\alpha - \sum_{i=1}^{k} \lambda_\alpha^i \varphi_i , \quad \varphi_\alpha = \sum_{i=1}^{k} \lambda_\alpha^i \varphi_i .
\]
We choose the \(\lambda_\alpha^i\)'s such that \(v_\alpha \in E_{\lambda_1}^\perp L^2\) (namely \(\lambda_\alpha^i = \int u_\alpha \varphi_i\)). We claim that
\[
\lim_{\alpha \to +\infty} (\lambda_\alpha - \lambda_1) \varphi_\alpha = 0 \quad \text{in} \quad C^{0,\theta} . \quad (P)
\]
We prove \((P)\). Since \((E')\) has a solution \(u_0 \neq 0\), integrating \((E')\) against \(\varphi \in E_{\lambda_1}\) there holds that \(f_0 \in E_{\lambda_1}^{\perp_{L^2}}\). Then, by \((E_\alpha)\),
\[
\int f_\alpha \varphi_i = \int (\Delta_g u_\alpha - \lambda_\alpha u_\alpha) \varphi_i
\]
\[
= \int u_\alpha (\Delta_g \varphi_i - \lambda_\alpha \varphi_i)
\]
\[
= (\lambda_1 - \lambda_\alpha) \int u_\alpha \varphi_i
\]
\[
= (\lambda_1 - \lambda_\alpha) \lambda_\alpha^i,
\]
and since \(f_\alpha \to f_0\) in \(C^{0,\theta}\), and \(f_0 \in E_{\lambda_1}^{\perp_{L^2}}\), we get that \((\lambda_1 - \lambda_\alpha) \lambda_\alpha^i \to 0\), and thus that \((\lambda_\alpha - \lambda_1) \varphi_\alpha \to 0\) smoothly. This proves \((P)\).

Now that we have \((P)\), we let \(\lambda_2 > \lambda_1\) be the second eigenvalue for \(\Delta_g\). By the variational characterisation of \(\lambda_2\),
\[
\lambda_2 \leq \frac{\int |\nabla v_\alpha|^2}{\int |v_\alpha - \overline{v_\alpha}|^2} \tag{(I)}
\]
for all \(\alpha\), where \(v_\alpha = u_\alpha - \varphi_\alpha\) is as above, and \(\overline{v_\alpha}\) is the average of \(v_\alpha\). The point here is that \(v_\alpha - \overline{v_\alpha}\) is \(L^2\)-orthogonal both to the constants and to \(E_{\lambda_1}\).
Since functions in $E_{\lambda_1}$ has zero average, we get from the definition of $v_\alpha$ that $\overline{v}_\alpha = \overline{u}_\alpha$. Then, by $(E_\alpha)$, $\overline{v}_\alpha = \overline{u}_\alpha = O(1)$. Still by $(E_\alpha)$ there holds that

$$\Delta_g v_\alpha - \lambda_\alpha v_\alpha = f_\alpha + (\lambda_\alpha - \lambda_1)\varphi_\alpha$$

$$(E'_\alpha)$$

for all $\alpha$. Then, by $(I)$ and $(E'_\alpha)$, using that $\overline{v}_\alpha = O(1)$ and that $\int(v_\alpha - \overline{v}_\alpha) = 0$, we get that

$$\int v_\alpha^2 = \int v_\alpha(v_\alpha - \overline{v}_\alpha) + O(1)$$

$$= \int(v_\alpha - \overline{v}_\alpha)^2 + O(1)$$

$$\leq \frac{1}{\lambda_2} \int |\nabla v_\alpha|^2 + O(1)$$

$$= \frac{\lambda_\alpha}{\lambda_2} \int v_\alpha^2 + \frac{1}{\lambda_2} \int f_\alpha v_\alpha + \frac{\lambda_\alpha - \lambda_1}{\lambda_2} \int \varphi_\alpha v_\alpha + O(1)$$

$$\leq \frac{\lambda_\alpha}{\lambda_2} \int v_\alpha^2 + O(\|v_\alpha\|_{L^2}) + O(1)$$

for all $\alpha$. Since $\lambda_\alpha \to \lambda_1$ and $\lambda_1 < \lambda_2$, it follows that $\|v_\alpha\|_{L^2} = O(1)$. Then, by $(E'_\alpha)$, and standard elliptic theory, since $(\lambda_\alpha - \lambda_1)\varphi_\alpha \to 0$ smoothly by $(P)$, we get that the $v_\alpha$'s are bounded in $H^1$ and that, up to a subsequence, $v_\alpha \to v$ in $C^2$, where $v$ solves $(E')$. 

Elliptic stability - Part II - An introduction to Elliptic Stability
Now, at this point, we let \( w = v - u_0 \), and
\[
w_{\alpha} = u_0 + w + \varphi_{\alpha}.
\]

There holds that \( w \in E_{\lambda_1} \) since \( u_0 \) and \( v \) both solve \((E')\). Since \( v_{\alpha} \to v \) in \( C^2 \), and \( v_{\alpha} = u_{\alpha} - \varphi_{\alpha} \), we get that \( u_{\alpha} - \varphi_{\alpha} \to u_0 + w \) in \( C^2 \) (note that \( v = u_0 + w \)), and thus that
\[
\| u_{\alpha} - w_{\alpha} \|_{C^2} \to 0 \quad (\ast\star)
\]
as \( \alpha \to +\infty \) (since \( u_{\alpha} - w_{\alpha} = u_{\alpha} - \varphi_{\alpha} - u_0 - w \)). There holds that
\[
\Delta_g w_{\alpha} - \lambda_1 w_{\alpha} = f_0 \quad (\ast\ast\ast)
\]
for all \( \alpha \), since \( w, \varphi_{\alpha} \in E_{\lambda_1} \) and \( u_0 \) solve \((E')\). Therefore, by \((\ast\star)\) and \((\ast\ast\ast)\),
\[
d_{C^2}(u_{\alpha}; S(\lambda_1, f_0)) \to 0
\]
as \( \alpha \to +\infty \), and this contradicts the \((\ast)\) contradiction assumption that
\[
d_{C^2}(u_{\alpha}; S(\lambda_1, f_0)) \geq \frac{\epsilon_0}{2}.
\]
This ends the proof of Claim 2.

By Claims 1 and 2, \((E')\) is geometrically stable but not analytically stable. Q.E.D.
II.4) The subcritical world:

Let \((M, g)\) smooth compact, \(\partial M = \emptyset\), \(n \geq 3\), and consider our nonlinear model equation in the subcritical setting. Namely,

\[
\Delta_g u + hu = u^{p-1}, \quad (E_h)
\]

\(u \geq 0\), \(p \in (2, 2^*)\). When \(h\) is such that \(\Delta_g + h\) is coercive, \((E_h)\) possesses a nontrivial (minimal) solution. Conversely, if \((E_h)\) has a nontrivial solution, then \(\Delta_g + h\) is coercive.

We perturb \((E_h)\) with respect to \(h\), e.g. in Hölder spaces \(C^{0, \theta}\), \(\theta \in (0, 1)\), and say for short that \((E_h)\) is analytically stable if for any sequences \((h_\alpha)_\alpha\) in \(C^{0, \theta}\), and \((u_\alpha)_\alpha\) in \(C^2\), satisfying that

\[
\begin{cases}
\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{p-1} \text{ for all } \alpha, \\
u_\alpha \geq 0 \text{ in } M \text{ for all } \alpha, \\
h_\alpha \to h \text{ in } C^{0, \theta} \text{ as } \alpha \to +\infty ,
\end{cases}
\]

there holds that, up to a subsequence, \(u_\alpha \to u\) in \(C^2\) for some solution \(u\) of \((E_h)\). This is the analytic stability notion we defined above, for nonnegative solutions, a set \(\mathcal{P}\) of \(\tilde{f}\) given by \(\tilde{f}(\cdot, u) = u^{p-1} - \tilde{h}(\cdot)u\), with \(\tilde{h} \in C^{0, \theta}\), and \(\|\tilde{f}\|_\mathcal{P} = \|\tilde{h}\|_{C^{0, \theta}}\). Then:
Theorem: (Subcritical stability, Gidas-Spruck, 81)

For any closed manifold \((M, g), n \geq 3\), and any \(h \in C^{0, \theta}\) such that \(\Delta_g + h\) is coercive, \((E_h)\) is analytically stable.

Proof (Baby blow-up theory): By contradiction, there exist \((h_\alpha)_\alpha\) and \((u_\alpha)_\alpha\) s.t.

\[
\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{p-1} \quad (E_{h_\alpha})
\]

in \(M\) for all \(\alpha\), the \(h_\alpha\)'s converge, and \(||u_\alpha||_{L^\infty} \to +\infty\). Let \(x_\alpha\) be s.t. \(u_\alpha(x_\alpha) = \max_M u_\alpha\). Let \(\mu_\alpha = ||u_\alpha||_{L^\infty}^{-(p-2)/2}\). Then \(\mu_\alpha \to 0\). Define

\[
\tilde{u}_\alpha(x) = \mu_\alpha^{2/(p-2)} u_\alpha \left( \exp_{x_\alpha} (\mu_\alpha x) \right),
\]

where \(x \in \mathbb{R}^n\). By construction, \(\tilde{u}_\alpha(0) = 1\) and \(0 \leq \tilde{u}_\alpha \leq 1\) for all \(\alpha\). Then

\[
\Delta_{\tilde{g}_\alpha} \tilde{u}_\alpha + \mu_\alpha^2 \tilde{h}_\alpha \tilde{u}_\alpha = \tilde{u}_\alpha^{p-1}, \quad (\tilde{E}_{h_\alpha})
\]

where \(\tilde{g}_\alpha(x) = \left( \exp_{x_\alpha}^* g \right)(\mu_\alpha x)\), and \(\tilde{h}_\alpha(x) = h_\alpha \left( \exp_{x_\alpha} (\mu_\alpha x) \right)\). There holds \(\tilde{g}_\alpha \to \delta\) in \(C^2_{loc}(\mathbb{R}^n)\). Since \(||\tilde{u}_\alpha||_{L^\infty} \leq 1\), standard elliptic theory \(\Rightarrow\) the \(\tilde{u}_\alpha\)'s converge in \(C^2_{loc}(\mathbb{R}^n)\). Let \(\tilde{u}\) be their limit. Then \(\Delta \tilde{u} = \tilde{u}^{p-1}\). By construction \(\tilde{u}(0) = 1\). And we get a contradiction with the Liouville theorem of Gidas and Spruck: the equation \(\Delta u = u^{p-1}\) doesn't have nonnegative nontrivial solutions in \(\mathbb{R}^n\) when \(p < 2^*\). Q.E.D.
II.5) More precise definitions are needed in the critical world:

Let \((M, g)\) closed, \(n \geq 3\). For \(k \in \mathbb{N}\), and \(\theta \in [0, 1]\), we adopt the convention that \(C^{k,0} = C^{k}\). Given \(h \in C^{k,\theta}\), we consider our model equation in the critical case

\[
\Delta_g u + hu = u^{2^*-1}, \quad (E_h)
\]

\(u \geq 0\), and we plan to perturb \((E_h)\) with respect to \(h\) in \(C^{k,\theta}\) (as in the subcritical case).

We adopt here the more refined following terminology by splitting analytic stability into three notions of analytic stability involving energy. We define:

- \(C^{k,\theta}\)-analytic \(\Lambda\)-stability,
- \(C^{k,\theta}\)-analytic stability,
- \(C^{k,\theta}\)-bounded stability,

by playing with the energy

\[E(u) = \int_M |u|^{2^*} \, dv_g\]

which, for solutions \(u\) of equations like \((E_h)\), turns out to be equivalent to \(\|u\|^2_{H^1}\).

As in the subcritical case, the existence of a nontrivial solution \(u \geq 0\) to \((E_h)\) implies that \(\Delta_g + h\) is coercive (a natural assumption we will face several time in the forthcoming slides).
Definition: (Analytic stability in the critical case)

Let $\Lambda > 0$. Equation $(E_h)$ is $C^{k,\theta}$-analytically $\Lambda$-stable if for any sequence $(h_\alpha)_\alpha$ in $C^{k,\theta}$ such that $h_\alpha \to h$ in $C^{k,\theta}$ as $\alpha \to +\infty$, and any sequence $(u_\alpha)_\alpha$, $u_\alpha \geq 0$, such that

$$\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{2*-1}$$

(E$_{h_\alpha}$)

in $M$ for all $\alpha$, satisfying that $\int_M u_\alpha^{2*} \, dv_g \leq \Lambda$ for all $\alpha$, there holds that, up to a subsequence, $u_\alpha \to u$ in $C^2$ as $\alpha \to +\infty$ for some solution $u$ of $(E_h)$. Equation $(E_h)$ is $C^{k,\theta}$-analytically stable if it is $C^{k,\theta}$-analytically $\Lambda$-stable for all $\Lambda > 0$. Equation $(E_h)$ is $C^{k,\theta}$-bounded and stable if it is $C^{k,\theta}$-analytically $\infty$-stable.

This definition has a natural companion dealing with compactness.

Definition: (Compactness)

Let $\Lambda > 0$. Equation $(E_h)$ is $\Lambda$-compact if any sequence $(u_\alpha)_\alpha$, $u_\alpha \geq 0$, of solutions of $(E_h)$ satisfying that $\int_M u_\alpha^{2*} \, dv_g \leq \Lambda$ for all $\alpha$, has a subsequence which converges in $C^2$ to a solution of $(E_h)$. Equation $(E_h)$ is compact if it is $\Lambda$-compact for all $\Lambda > 0$. Equation $(E_h)$ is bounded and compact if it is $\infty$-compact.
Rk1 : The analytic stability notions are ordered (bounded stability $\Rightarrow$ analytic stability $\Rightarrow$ analytic $\Lambda$-stability for all $\Lambda > 0$) and the more we increase $k$, the less we actually demand ($C^{k',\theta}$-stability $\Rightarrow$ $C^{k,\theta}$-stability if $k' \leq k$).

Rk2 : We have that stability $\Rightarrow$ compactness ($C^{k,\theta}$-bounded stability $\Rightarrow$ bounded compactness, $C^{k,\theta}$-analytic stability $\Rightarrow$ compactness, $C^{k,\theta}$-analytic $\Lambda$-stability $\Rightarrow$ $\Lambda$-compactness for all $\Lambda > 0$, for all $k$ and $\theta$).

The difference between stability and compactness turns out be precisely the notion of geometric stability that we discussed in II.3, and we have that Analytic stability = Geometric stability + Compactness.


Let $k \in \mathbb{N}$, $\theta \in [0, 1]$, and $\Lambda > 0$. Equation $(E_h)$ is $C^{k,\theta}$-analytically $\Lambda$-stable if and only if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall h \in C^{k,\theta}, \| \tilde{h} - h \|_{C^{k,\theta}} \Rightarrow d_{C^2}(S^\Lambda_{\tilde{h}}; S^\Lambda_h) < \varepsilon \quad (GS)$$

and $(E_h)$ is $\Lambda$-compact, where $S^\Lambda_h$ is the set of the solutions $u$ of $(E_{\tilde{h}})$ which satisfy that $E(u) \leq \Lambda$. 
Proof of the Proposition: The implication “Analyt.Stab. ⇒ Geom.Stab. + Cptness” is obvious. Conversely, we assume (GS) and that \((E_h)\) is \(\Lambda\)-compact. Let \((h_\alpha)_\alpha\) be a sequence in \(C^{k,\theta}\) such that \(h_\alpha \to h\) in \(C^{k,\theta}\). Let also \((u_\alpha)_\alpha\) be such that the \(u_\alpha\)'s solve \((E_{h_\alpha})\) and satisfy that \(E(u_\alpha) \leq \Lambda\) for all \(\alpha\). By (GS) there exists a sequence \((v_\alpha)_\alpha\) in \(S^\Lambda_h\) such that \(\|v_\alpha - u_\alpha\|_{C^2} \to 0\) as \(\alpha \to +\infty\). By the \(\Lambda\)-compactness of \((E_h)\), since the \(v_\alpha\)'s are all in \(S^\Lambda_h\), we also have that there exists \(v \in S^\Lambda_h\) such that, up to a subsequence, \(v_\alpha \to v\) in \(C^2\) as \(\alpha \to +\infty\). Then we clearly get that, up to a subsequence, \(u_\alpha \to v\) in \(C^2\) as \(\alpha \to +\infty\), and this proves the \(C^{k,\theta}\)-analytic \(\Lambda\)-stability of \((E_h)\). Q.E.D.

Anticipating on what we are going to discuss in Part IV, the following proposition holds true.

**Proposition : (Compactness \(\not\Rightarrow\) Analytic Stability)**

There are equations like \((E_h)\) which are compact but unstable.

There are sophisticated examples of such a fact, but also very easy examples like the Yamabe equation in the projective space \(\mathbb{P}^n(\mathbb{R})\) when \(n \geq 6\). As proved in II.4, the situation described in the proposition does not occur in the subcritical case of \((E_h)\).
Thank you for your attention!