Elliptic stability for stationary Schrödinger equations

by

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Part II/VI

An introduction to elliptic stability

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PART II. AN INTRODUCTION TO ELLIPTIC STABILITY.

II.1) The model equation.

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NOTE : The blue writing is what you have to write down to be able to follow the slides presentation.

PART II. AN INTRODUCTION TO ELLIPTIC STABILITY.

II.1) The model equation :

(M, g) smooth compact, $\partial M = \emptyset$ (closed manifold), $n \geq 3$.

Model equation

$$\Delta_g u + hu = u^{p-1} \qquad (E_h)$$

Varying h's

Here : $u \geq 0$, $\Delta_g = -\mathrm{div}_g \nabla$, $h \in C^{0,\theta}$ (typically), $p \in (2, 2^\star]$, where $2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent. H^1 Sobolev space of functions in L^2 with one derivative in L^2. Then $H^1 \subset L^p$ for all $p \leq 2^\star$, and

$H^1 \subset L^p$ is compact when $p < 2^\star$,
but not when $p = 2^\star$.

$$\swarrow \qquad\qquad\qquad \searrow$$

Subcritical "world" $p < 2^\star$	$\neq$	Critical "world" $p = 2^\star$

Question : How much is (E_h) robust with respect to h ?

II.2) Equations behind the model equation :

- The Yamabe equation
- The stationary Klein-Gordon-Maxwell-Proca system
- The Einstein-Lichnerowicz equation
- The Kirchhoff equation

The Yamabe equation is obviously of the (E_h)-type. It is written as

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = u^{2^\star - 1} . \qquad (Y)$$

We get an equation like (E_h) , where

$$h = \frac{n-2}{4(n-1)} S_g$$

is given by the geometry of the manifold (and $p = 2^\star$ is critical). As we saw, the LHS in (Y) is the conformal Laplacian (and it enjoys conformal invariance).

The stationary Klein-Gordon-Maxwell-Proca system in reduced form is also of the (E_h)-type. The $(KGMP_r)$-system is written as

$$\begin{cases} \Delta_g u + m_0^2 u = u^{2^\star-1} + \omega^2(qv-1)^2 u \\ \Delta_g v + \left(m_1^2 + q^2u^2\right) v = qu^2 \ . \end{cases} \qquad (KGMP_r)$$

We can always solve the equation

$$\Delta_g \Phi(u) + (m_1^2 + q^2u^2)\Phi(u) = qu^2$$

and we get a map $\Phi : H^1 \to H^1$. Then the $(KGMP_r)$-system reduces to the first equation

$$\Delta_g u + m_0^2 u = u^{2^\star-1} + \omega^2 \left(q\Phi(u) - 1\right)^2 u \ .$$

The solutions of the system are the couples $(u, \Phi(u))$. We get an equation like (E_h), where h is now given by

$$h = m_0^2 - \omega^2(q\Phi(u) - 1)^2 \ .$$

In particular, h depends on u, and (in the 3d-model) h turns out to be controlled in $C^{0,\theta}$-topologies.

The Einstein-scalar field Lichnerowicz equation corresponds to the Hamiltonian constraint in the constraint equations in the conformal method setting (Lichnerowicz). The two constraint equations

(Hamiltonian + Momentum) are written (conformal method setting) as :

$$\begin{cases} \Delta_g u + h_0 u = f u^{2^\star - 1} + \frac{a}{u^{2^\star+1}} & \text{(EL)} \\ \overrightarrow{\Delta}_g X = \frac{n-1}{n} u^{2^\star} \nabla\tau - \pi\nabla\psi & \text{(MC)} \end{cases}$$

where h_0, f and a are given (depending on the geometry and physics data), u is an unknown function, X is an unknown vector field, and $\overrightarrow{\Delta}_g = \nabla.\mathcal{L}$ ($\mathcal{L}$ the conformal Killing operator). The (EL)-equation is the Einstein-Lichnerowicz equation. It is highly nonlinear and, in the CMC-case (where $\tau = C^{st}$) it fully describes the (CE)-system, since then the two equations are independent (and (MC) is a "basic" Laplace type equation).

The negative power term in (EL) $\Rightarrow$ there exists $\varepsilon_0 > 0$ s.t. $u \geq \varepsilon_0$ for all solution of the Hamiltonian constraint. A very basic argument when $a > 0$ is as follows : let x_0 be a point where u is minimum. Then $\Delta_g u(x_0) \leq 0$ and we get from (EL) that

$$\frac{a(x_0)}{u(x_0)^{2^\star+1}} + f(x_0)u(x_0)^{2^\star-1} \leq h_0(x_0)u(x_0)$$

and when $a > 0$ this obviously implies that there exists $\varepsilon_0 > 0$, independent of u, such that $u \geq \varepsilon_0$ in M.

In specific cases $f \equiv 1$. Then we recover an equation like (E_h), where

$$h = h_0 - \frac{a}{u^{2^\star+2}} \ .$$

In particular h depends again on u, and h is here controlled in the L^∞-topology.

The Kirchhoff equation is written as

$$\left(a + b\int_M |\nabla u|^2 dv_g\right)\Delta_g u + h_0 u = u^{2^\star-1} \ , \qquad (K)$$

where $a, b > 0$ are positive real numbers and $h_0 \in C^1(M,\mathbb{R})$. Let $K(u) = a + b\int_M |\nabla u|^2 dv_g$, and define $v = K(u)^{-\frac{1}{2^\star-2}}u$. Then

$$\Delta_g v + hv = v^{2^\star-1} \ ,$$

and we recover an equation like (E_h), where $h = \frac{h_0}{K(u)}$. Here again h depends on u, and h is in this case controlled in the C^1-topology.

Moral : There are several models hidden in our model equation (E_h) when h depends on the solution u. The sole control on the set in which h varies will have to matter in our approach.

II.3) A first insight into elliptic stability :

Consider equations like

$$\Delta_g u = f(x, u)\ , \qquad (E)$$

where $f : M \times \mathbb{R} \to \mathbb{R}$ is given, and the Laplacian $\Delta_g = -\mathrm{div}_g \nabla$ is the Laplace-Beltrami operator.

Goal : define the stability (robustness) of (E) with respect to f .

Let S_f be the set of solutions of (E). Let $\mathcal{P}$ be a set of perturbations of f, namely a family of functions $\tilde{f} : M \times \mathbb{R} \to \mathbb{R}$ such that $f \in \mathcal{P}$. For the sake of simplicity we assume $S_{\tilde{f}} \subset C^2$ for all $\tilde{f} \in \mathcal{P}$. Define the pointed distance between subsets of C^2 by

$$d_{C^2}^{\hookrightarrow}(X; Y) = \sup_{v \in X} \inf_{u \in Y} \|v - u\|_{C^2}\ ,$$

and we adopt the conventions that $d_{C^2}^{\hookrightarrow}(X; \emptyset) = +\infty$ if $X \neq \emptyset$, and $d_{C^2}^{\hookrightarrow}(\emptyset; Y) = 0$ for all Y. Then, $d_{C^2}^{\hookrightarrow}(X; Y) = 0$ iff $X \subset \overline{Y}$, and $d_{C^2}^{\hookrightarrow}$ satisfies the triangle inequality

$$d_{C^2}^{\hookrightarrow}(X; Z) \leq d_{C^2}^{\hookrightarrow}(X; Y) + d_{C^2}^{\hookrightarrow}(Y; Z)$$

for all $X, Y, Z \subset C^2$.

We consider

$$\Delta_g u = f(x, u) \ , \qquad (E)$$

and define two notions of stability for (E).

Definition : (Geometric and Analytic stability)

Equation (E) is geometrically stable with respect to a set $\mathcal{P}$ of perturbations of f and a norm $\|\cdot\|_{\mathcal{P}}$ on $\mathcal{P}$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall \tilde{f} \in \mathcal{P}, \|\tilde{f} - f\|_{\mathcal{P}} < \delta \ \Rightarrow \ d_{C^2}^{\hookrightarrow}(S_{\tilde{f}}; S_f) < \varepsilon \ ;$$

Equation (E) is analytically stable with respect to $\mathcal{P}$ and $\|\cdot\|_{\mathcal{P}}$ if for any sequence $(f_\alpha)_\alpha$ in $\mathcal{P}$, converging to f w.r.t. $\|\cdot\|_{\mathcal{P}}$ as $\alpha \to +\infty$, and any sequence $(u_\alpha)_\alpha$ of solutions of $\Delta_g u_\alpha = f_\alpha(\cdot, u_\alpha)$ in M, there holds that, up to a subsequence, $u_\alpha \to u$ in C^2 as $\alpha \to +\infty$, where u solves (E).

Geometric stability expresses the fact that S_f is stable with respect to perturbations of f. It corresponds to the continuity in $\mathcal{P}$ of the function $\tilde{f} \to d_{C^2}^{\hookrightarrow}(S_{\tilde{f}}; S_f)$. It is easily checked (by contradiction) that :

$$\text{Analytic stability} \ \Rightarrow \ \text{Geometric stability} \ .$$

The converse is false in general as we can prove below.

An example of a geometrically stable equation which turns out to be not analytically stable : Let $\lambda_1 \in \mathrm{Sp}(\Delta_g)$ be the first nonzero eigenvalue of Δ_g, $\lambda_1 > 0$. Let $u_0 \not\equiv 0$ and $f_0 \not\equiv 0$ be smooth functions satisfying that $\Delta_g u_0 - \lambda_1 u_0 = f_0$, and consider the equation

$$\Delta_g u - \lambda_1 u = f_0 \ . \tag{E'}$$

Then u_0 solves (E'). We let $\mathcal{P} = \left\{ \tilde{f}(\cdot, u) = f(\cdot) + \lambda u, \lambda \in \mathbb{R}, f \in C^{0,\theta} \right\}$, and define $\| \cdot \|_{\mathcal{P}}$ by

$$\|\tilde{f}\|_{\mathcal{P}} = |\lambda| + \|f\|_{C^{0,\theta}} \ .$$

In other words, we perturb (E') by perturbing λ_1 and f_0 in $\mathbb{R} \times C^{0,\theta}$.

Claim 1 : (E') is not analytically stable (and not even compact). We see this by picking $\varphi \not\equiv 0$ in the eigenspace associated to λ_1. We let $(k_\alpha)_\alpha$ be a sequence of positive real numbers s.t. $k_\alpha \to +\infty$ as $\alpha \to +\infty$. We define

$$u_\alpha = u_0 + k_\alpha \varphi \ .$$

Obviously, the u_α's all solve (E'). However $\|u_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$, and this contradicts the analytic stability of (E').

Claim 2 : We claim that (E') is geometrically stable (w.r.t. perturbations of λ_1 and f_0 in $\mathbb{R}\times C^{0,\theta}$). We prove this by contradiction. Then there exists $\varepsilon_0>0$, a sequence $(\lambda_\alpha)_\alpha\in\mathbb{R}$ such that $\lambda_\alpha\to\lambda_1$ as $\alpha\to+\infty$, and a sequence $(f_\alpha)_\alpha\in C^{0,\theta}$ such that $f_\alpha\to f_0$ in $C^{0,\theta}$ as $\alpha\to+\infty$, with the property that

$$d^{\hookrightarrow}_{C^2}\left(S_{(\lambda_\alpha,f_\alpha)};S_{(\lambda_1,f_0)}\right)\geq\varepsilon_0\ , \tag{$\star$}$$

where $S_{(\lambda,f)}$ stands for the set of solutions of $\Delta_g u-\lambda u=f$ (so that $S_{(\lambda_1,f_0)}$ is precisely the set of solutions of (E')). In particular, it follows from $(\star)$ that there exists a sequence $(u_\alpha)_\alpha$ of C^2-functions such that

$$\Delta_g u_\alpha-\lambda_\alpha u_\alpha=f_\alpha \tag{E_α}$$

for all α, and such that $d_{C^2}(u_\alpha;S_{(\lambda_1,f_0)})\geq\frac{\varepsilon_0}{2}$ for all α.Let E_{λ_1} be the eigenspace of Δ_g associated to λ_1. We know E_{λ_1} is finite dimensional. We let $\varphi_1,\dots,\varphi_k$ be a L^2-orthonormal basis for E_{λ_1}, and let v_α and φ_α be given by

$$v_\alpha=u_\alpha-\sum_{i=1}^k\lambda_\alpha^i\varphi_i\ ,\ \varphi_\alpha=\sum_{i=1}^k\lambda_\alpha^i\varphi_i\ .$$

We choose the λ_α^i's such that $v_\alpha\in E_{\lambda_1}^{\perp_{L^2}}$ (namely $\lambda_\alpha^i=\int u_\alpha\varphi_i$). We claim that

$$\lim_{\alpha\to+\infty}(\lambda_\alpha-\lambda_1)\varphi_\alpha=0\ \text{ in }C^{0,\theta}. \tag{P}$$

We prove (P). Since (E') has a solution $u_0 \not\equiv 0$, integrating (E') against $\varphi \in E_{\lambda_1}$ there holds that $f_0 \in E_{\lambda_1}^{\perp_{L^2}}$. Then, by (E_α),

$$\begin{aligned}\int f_\alpha \varphi_i &= \int \left(\Delta_g u_\alpha - \lambda_\alpha u_\alpha\right) \varphi_i \\ &= \int u_\alpha \left(\Delta_g \varphi_i - \lambda_\alpha \varphi_i\right) \\ &= \left(\lambda_1 - \lambda_\alpha\right) \int u_\alpha \varphi_i \\ &= \left(\lambda_1 - \lambda_\alpha\right) \lambda_\alpha^i \ ,\end{aligned}$$

and since $f_\alpha \to f_0$ in $C^{0,\theta}$, and $f_0 \in E_{\lambda_1}^{\perp_{L^2}}$, we get that $\left(\lambda_1 - \lambda_\alpha\right) \lambda_\alpha^i \to 0$, and thus that $(\lambda_\alpha - \lambda_1)\varphi_\alpha \to 0$ smoothly. This proves (P).

Now that we have (P), we let $\lambda_2 > \lambda_1$ be the second eigenvalue for Δ_g. By the variational characterisation of λ_2,

$$\lambda_2 \leq \frac{\int |\nabla v_\alpha|^2}{\int |v_\alpha - \overline{v}_\alpha|^2} \qquad (I)$$

for all α, where $v_\alpha = u_\alpha - \varphi_\alpha$ is as above, and $\overline{v}_\alpha$ is the average of v_α. The point here is that $v_\alpha - \overline{v}_\alpha$ is L^2-orthogonal both to the constants and to E_{λ_1}.

Since functions in E_{λ_1} has zero average, we get from the definition of v_α that $\overline{v}_\alpha = \overline{u}_\alpha$. Then, by (E_α), $\overline{v}_\alpha = \overline{u}_\alpha = O(1)$. Still by (E_α) there holds that

$$\Delta_g v_\alpha - \lambda_\alpha v_\alpha = f_\alpha + (\lambda_\alpha - \lambda_1)\varphi_\alpha \tag{E'_α}$$

for all α. Then, by (I) and (E'_α), using that $\overline{v}_\alpha = O(1)$ and that $\int (v_\alpha - \overline{v}_\alpha) = 0$, we get that

$$\begin{aligned}
\int v_\alpha^2 &= \int v_\alpha(v_\alpha - \overline{v}_\alpha) + O(1) \\
&= \int (v_\alpha - \overline{v}_\alpha)^2 + O(1) \\
&\le \frac{1}{\lambda_2} \int |\nabla v_\alpha|^2 + O(1) \\
&= \frac{\lambda_\alpha}{\lambda_2} \int v_\alpha^2 + \frac{1}{\lambda_2} \int f_\alpha v_\alpha + \frac{\lambda_\alpha - \lambda_1}{\lambda_2} \int \varphi_\alpha v_\alpha + O(1) \\
&\le \frac{\lambda_\alpha}{\lambda_2} \int v_\alpha^2 + O\left(\|v_\alpha\|_{L^2}\right) + O(1)
\end{aligned}$$

for all α. Since $\lambda_\alpha \to \lambda_1$ and $\lambda_1 < \lambda_2$, it follows that $\|v_\alpha\|_{L^2} = O(1)$. Then, by (E'_α), and standard elliptic theory, since $(\lambda_\alpha - \lambda_1)\varphi_\alpha \to 0$ smoothly by (P), we get that the v_α's are bounded in H^1 and that, up to a subsequence, $v_\alpha \to v$ in C^2, where v solves (E').

Now, at this point, we let $w = v - u_0$, and

$$w_\alpha = u_0 + w + \varphi_\alpha \ .$$

There holds that $w \in E_{\lambda_1}$ since u_0 and v both solve (E'). Since $v_\alpha \to v$ in C^2, and $v_\alpha = u_\alpha - \varphi_\alpha$, we get that $u_\alpha - \varphi_\alpha \to u_0 + w$ in C^2 (note that $v = u_0 + w$), and thus that

$$\|u_\alpha - w_\alpha\|_{C^2} \to 0 \qquad (\star\star)$$

as $\alpha \to +\infty$ (since $u_\alpha - w_\alpha = u_\alpha - \varphi_\alpha - u_0 - w$). There holds that

$$\Delta_g w_\alpha - \lambda_1 w_\alpha = f_0 \qquad (\star\star\star)$$

for all α, since $w, \varphi_\alpha \in E_{\lambda_1}$ and u_0 solve (E'). Therefore, by $(\star\star)$ and $(\star\star\star)$,

$$d_{C^2}(u_\alpha; S_{(\lambda_1, f_0)}) \to 0$$

as $\alpha \to +\infty$, and this contradicts the $(\star)$ contradiction assumption that $d_{C^2}(u_\alpha; S_{(\lambda_1, f_0)}) \geq \frac{\varepsilon_0}{2}$. This ends the proof of Claim 2.

By Claims 1 and 2, (E') is geometrically stable but not analytically stable. Q.E.D. ∎

II.4) The subcritical world :

Let (M, g) smooth compact, $\partial M = \emptyset$, $n \geq 3$, and consider our nonlinear model equation in the subcritical setting. Namely,

$$\Delta_g u + hu = u^{p-1} \ , \tag{E_h}$$

$u \geq 0$, $p \in (2, 2^\star)$. When h is such that $\Delta_g + h$ is coercive, (E_h) possesses a nontrivial (minimal) solution. Conversely, if (E_h) has a nontrivial solution, then $\Delta_g + h$ is coercive.

We perturb (E_h) with respect to h, e.g. in Hölder spaces $C^{0,\theta}$, $\theta \in (0, 1)$, and say for short that (E_h) is analytically stable if for any sequences $(h_\alpha)_\alpha$ in $C^{0,\theta}$, and $(u_\alpha)_\alpha$ in C^2, satisfying that

$$\begin{cases} \Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{p-1} \text{ for all } \alpha, \\ u_\alpha \geq 0 \text{ in } M \text{ for all } \alpha, \\ h_\alpha \to h \text{ in } C^{0,\theta} \text{ as } \alpha \to +\infty \ , \end{cases} \tag{E_α}$$

there holds that, up to a subsequence, $u_\alpha \to u$ in C^2 for some solution u of (E_h). This is the analytic stability notion we defined above, for nonnegative solutions, a set $\mathcal{P}$ of $\tilde{f}$ given by $\tilde{f}(\cdot, u) = u^{p-1} - \tilde{h}(\cdot)u$, with $\tilde{h} \in C^{0,\theta}$, and $\|\tilde{f}\|_{\mathcal{P}} = \|\tilde{h}\|_{C^{0,\theta}}$. Then :

Theorem : (Subcritical stability, Gidas-Spruck, 81)

For any closed manifold (M, g), $n \geq 3$, and any $h \in C^{0,\theta}$ such that $\Delta_g + h$ is coercive, (E_h) is analytically stable.

Proof (Baby blow-up theory) : By contradiction, there exist $(h_\alpha)_\alpha$ and $(u_\alpha)_\alpha$ s.t.

$$\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{p-1} \qquad (E_{h_\alpha})$$

in M for all α, the h_α's converge, and $\|u_\alpha\|_{L^\infty} \to +\infty$. Let x_α be s.t. $u_\alpha(x_\alpha) = \max_M u_\alpha$. Let $\mu_\alpha = \|u_\alpha\|_{L^\infty}^{-(p-2)/2}$. Then $\mu_\alpha \to 0$. Define

$$\tilde{u}_\alpha(x) = \mu_\alpha^{\frac{2}{p-2}} u_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right) \ ,$$

where $x \in \mathbb{R}^n$. By construction, $\tilde{u}_\alpha(0) = 1$ and $0 \leq \tilde{u}_\alpha \leq 1$ for all α. Then

$$\Delta_{\tilde{g}_\alpha} \tilde{u}_\alpha + \mu_\alpha^2 \tilde{h}_\alpha \tilde{u}_\alpha = \tilde{u}_\alpha^{p-1} \ , \qquad (\tilde{E}_{h_\alpha})$$

where $\tilde{g}_\alpha(x) = \left(\exp_{x_\alpha}^\star g\right)(\mu_\alpha x)$, and $\tilde{h}_\alpha(x) = h_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right)$. There holds $\tilde{g}_\alpha \to \delta$ in $C^2_{loc}(\mathbb{R}^n)$. Since $\|\tilde{u}_\alpha\|_{L^\infty} \leq 1$, standard elliptic theory $\Rightarrow$ the $\tilde{u}_\alpha$'s converge in $C^2_{loc}(\mathbb{R}^n)$. Let $\tilde{u}$ be their limit. Then $\Delta \tilde{u} = \tilde{u}^{p-1}$. By construction $\tilde{u}(0) = 1$. And we get a contradiction with the Liouville theorem of Gidas and Spruck : the equation $\Delta u = u^{p-1}$ doesn't have nonnegative nontrivial solutions in $\mathbb{R}^n$ when $p < 2^\star$. Q.E.D. ∎

II.5) More precise definitions are needed in the critical world :

Let (M, g) closed, $n \geq 3$. For $k \in \mathbb{N}$, and $\theta \in [0, 1]$, we adopt the convention that $C^{k,0} = C^k$. Given $h \in C^{k,\theta}$, we consider our model equation in the critical case

$$\Delta_g u + hu = u^{2^\star - 1} \ , \qquad (E_h)$$

$u \geq 0$, and we plan to perturb (E_h) with respect to h in $C^{k,\theta}$ (as in the subcritical case).

We adopt here the more refined following terminology by splitting analytic stability into three notions of analytic stability involving energy. We define :

- $C^{k,\theta}$-analytic Λ-stability,
- $C^{k,\theta}$-analytic stability,
- $C^{k,\theta}$-bounded stability,

by playing with the energy $E(u) = \int_M |u|^{2^\star} dv_g$ which, for solutions u of equations like (E_h), turns out to be equivalent to $\|u\|^2_{H^1}$.

As in the subcritical case, the existence of a nontrivial solution $u \geq 0$ to (E_h) implies that $\Delta_g + h$ is coercive (a natural assumption we will face several time in the forthcoming slides).

Definition : (Analytic stability in the critical case)

Let $\Lambda > 0$. Equation (E_h) is $C^{k,\theta}$**-analytically** Λ**-stable** if for any sequence $(h_\alpha)_\alpha$ in $C^{k,\theta}$ such that $h_\alpha \to h$ in $C^{k,\theta}$ as $\alpha \to +\infty$, and any sequence $(u_\alpha)_\alpha$, $u_\alpha \geq 0$, such that

$$\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{2^\star - 1} \qquad (E_{h_\alpha})$$

in M for all α, satisfying that $\int_M u_\alpha^{2^\star} dv_g \leq \Lambda$ for all α, there holds that, up to a subsequence, $u_\alpha \to u$ in C^2 as $\alpha \to +\infty$ for some solution u of (E_h). Equation (E_h) is $C^{k,\theta}$**-analytically stable** if it is $C^{k,\theta}$-analytically Λ-stable for all $\Lambda > 0$. Equation (E_h) is $C^{k,\theta}$**-bounded and stable** if it is $C^{k,\theta}$-analytically ∞-stable.

This definition has a natural companion dealing with compactness.

Definition : (Compactness)

Let $\Lambda > 0$. Equation (E_h) is Λ-**compact** if any sequence $(u_\alpha)_\alpha$, $u_\alpha \geq 0$, of solutions of (E_h) satisfying that $\int_M u_\alpha^{2^\star} dv_g \leq \Lambda$ for all α, has a subsequence which converges in C^2 to a solution of (E_h). Equation (E_h) is **compact** if it is Λ-compact for all $\Lambda > 0$. Equation (E_h) is **bounded and compact** if it is ∞-compact.

Rk1 : The analytic stability notions are ordered (bounded stability $\Rightarrow$ analytic stability $\Rightarrow$ analytic Λ-stability for all $\Lambda > 0$) and the more we increase k, the less we actually demand ($C^{k',\theta}$-stability $\Rightarrow$ $C^{k,\theta}$-stability if $k' \leq k$).

Rk2 : We have that stability $\Rightarrow$ compactness ($C^{k,\theta}$-bounded stability $\Rightarrow$ bounded compactness, $C^{k,\theta}$-analytic stability $\Rightarrow$ compactness, $C^{k,\theta}$-analytic Λ-stability $\Rightarrow$ Λ-compactness for all $\Lambda > 0$, for all k and θ).

The difference between stability and compactness turns out be precisely the notion of geometric stability that we discussed in II.3, and we have that Analytic stability $=$ Geometric stability $+$ Compactness.

Proposition : (Analyt.Stab. $=$ Geom.Stab. $+$ Cptness)

Let $k \in \mathbb{N}$, $\theta \in [0,1]$, and $\Lambda > 0$. Equation (E_h) is $C^{k,\theta}$-analytically Λ-stable if and only if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall \tilde{h} \in C^{k,\theta},\ \|\tilde{h} - h\|_{C^{k,\theta}} \ \Rightarrow\ d^{\hookleftarrow}_{C^2}\left(S^{\Lambda}_{\tilde{h}}; S^{\Lambda}_{h}\right) < \varepsilon \quad (GS)$$

and (E_h) is Λ-compact, where $S^{\Lambda}_{\tilde{h}}$ is the set of the solutions u of $(E_{\tilde{h}})$ which satisfy that $E(u) \leq \Lambda$.

Proof of the Proposition : The implication "Analyt.Stab. $\Rightarrow$ Geom.Stab. + Cptness" is obvious. Conversely, we assume (GS) and that (E_h) is Λ-compact. Let $(h_\alpha)_\alpha$ be a sequence in $C^{k,\theta}$ such that $h_\alpha \to h$ in $C^{k,\theta}$. Let also $(u_\alpha)_\alpha$ be such that the u_α's solve (E_{h_α}) and satisfy that $E(u_\alpha) \leq \Lambda$ for all α. By (GS) there exists a sequence $(v_\alpha)_\alpha$ in S_h^Λ such that $\|v_\alpha - u_\alpha\|_{C^2} \to 0$ as $\alpha \to +\infty$. By the Λ-compactness of (E_h), since the v_α's are all in S_h^Λ, we also have that there exists $v \in S_h^\Lambda$ such that, up to a subsequence, $v_\alpha \to v$ in C^2 as $\alpha \to +\infty$. Then we clearly get that, up to a subsequence, $u_\alpha \to v$ in C^2 as $\alpha \to +\infty$, and this proves the $C^{k,\theta}$-analytic Λ-stability of (E_h). Q.E.D. ∎

Anticipating on what we are going to discuss in Part IV, the following proposition holds true.

Proposition : (Compactness $\not\Rightarrow$ Analytic Stability)

There are equations like (E_h) which are compact but unstable.

There are sophisticated examples of such a fact, but also very easy examples like the Yamabe equation in the projective space $\mathbb{P}^n(\mathbb{R})$ when $n \geq 6$. As proved in II.4, the situation described in the proposition does not occur in the subcritical case of (E_h).

Thank you for your attention !