Schrödinger-Proca Constructions in the closed setting

by

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Trends in Calculus of Variations and PDEs
University of Sussex and Ghent University
Online conference
May 2022

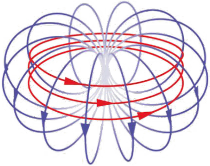
I. The equations

We are here going to discuss two systems of equations :

1- The Schrödinger-Poisson-Proca system,

2- The Bopp-Podolsky-Schrödinger-Proca system.

They both belong to those theories where the goal is to provide a model for the interaction between a charged non-relativistic quantum mechanical particle and the electromagnetic field that it generates. There, the electromagnetic field is both generated by and drives the particle field.



The Schrödinger-Poisson-Proca system is 2nd order. The Bopp-Podolsky-Schrödinger-Proca system is 4th order.

The background is a 3-dimensional closed Riemannian manifold (M, g). The Bopp-Podolsky-Schrödinger-Proca system we are going to discuss is given by

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \Phi(x, v, A)u = u^{p-1} \\ a^2\Delta_g^2 v + \Delta_g v + m_1^2 v = 4\pi q u^2 \\ a^2\Delta_g^2 A + \Delta_g A + m_1^2 A = \frac{4\pi q\hbar}{m_0^2}\Psi(A, S)u^2 \ . \end{cases} \qquad (BPSP)_a$$

The unknowns are (u, v, A), where u and v are functions, $u \geq 0$ in M, and A is a 1-form (u corresponds to the amplitude of the particle field that we write in polar form, (v, A) represents the electromagnetic field that the particle field creates). The whole system corresponds to an electro-magneto-static regime. Here :

$$\Phi(x, v, A) = \frac{\hbar^2}{2m_0^2}|\Psi(A, S)|^2 + \omega^2 + qv \ ,$$

and $\Psi(A, S) = \nabla S - \frac{q}{\hbar}A$. Here S is a given function. Then $q, m_0, m_1 > 0$ and $\omega \in \mathbb{R}$. The Bopp-Podolsky parameter is $a \geq 0$, $\hbar$ is the reduced Planck's constant and $p \in (2, 6]$.

In a similar way, the Schrödinger-Poisson-Proca system we are going to discuss is given by

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \Phi(x,v,A)u = u^{p-1} \\ \Delta_g v + m_1^2 v = 4\pi q u^2 \\ \Delta_g A + m_1^2 A = \frac{4\pi q\hbar}{m_0^2}\Psi(A,S)u^2 \ . \end{cases} \qquad (SPP)$$

Here again the unkowns are $(u, v, A) =$ (function,function,1-form) and we require that $u \geq 0$. As before,

$$\Phi(x,v,A) = \frac{\hbar^2}{2m_0^2}|\Psi(A,S)|^2 + \omega^2 + qv \ ,$$

$$\Psi(A,S) = \nabla S - \frac{q}{\hbar}A \ .$$

Then we get that

$$(SPP) = (BPSP)_a \text{ when } a = 0 \ .$$

In these equations $\Delta_g = -\text{div}_g\nabla$ is the Laplace-Beltrami operator (for functions), or $\Delta_g = \delta d + d\delta$ is the Hodge-de Rham Laplacian (for 1-forms), d the differential, $\delta = -\nabla.$ the codifferential.

Part 1

Construction of the equations, meaning, history

II. The larger equations

These two equations we are going to discuss are parts of a larger picture. We use Lagrangian constructions. The particle field is here represented by a function ψ and the electromagnetic field is represented by a gauge potential (A, φ), where φ (a function) represents the electric field and A (a 1-form) represents the magnetic field that the particle field creates.

Particle field $= \psi$,
Electromagnetic field $= (A, \varphi)$,

We adopt here the m_1-Proca formalism meaning that a mass is given to the electromagnetic field (φ, A). The particle field ψ is ruled by a nonlinear Schrödinger equation. The electromagnetic field (φ, A) is ruled by the Bopp-Podolsky-Proca action in the Bopp-Podolsky-Proca model. We need then to couple the Schrödinger and the Bopp-Podolsky-Proca actions.

Coupling is done by using the minimum coupling rule

$$\partial_t \to \tilde{\partial}_t = \partial_t + i\frac{q}{\hbar}\varphi \ , \ \ \nabla \to \tilde{\nabla} = \nabla - i\frac{q}{\hbar}A \ , \tag{1}$$

where q is the charge of ψ. The minimum coupling rule is the rule used in electrodynamics to account for all electromagnetic interactions.

The nonlinear Schrödinger Lagrangian (focusing case, i.e. the nonlinearity competes with the Laplacian) for ψ is then given by

$$\begin{aligned} \mathcal{L}_{NLS} &= i\hbar\frac{\partial\psi}{\partial t}\overline{\psi} - q\varphi|\psi|^2 - \frac{\hbar^2}{2m_0^2}|\nabla\psi - i\frac{q}{\hbar}A\psi|^2 + \frac{2}{p}|\psi|^p \\ &= i\hbar\frac{\tilde{\partial}\psi}{\partial t}\overline{\psi} - \frac{\hbar^2}{2m_0^2}|\tilde{\nabla}\psi|^2 + \frac{2}{p}|\psi|^p \ . \end{aligned} \tag{2}$$

This is nothing but the usual nonlinear Schrödinger Lagrangian when time and space derivative are given by the coupling (1). It governs ψ.

It remains to write down the Lagrangian which is going to govern (A,φ). We assume in this section (and essentially only in this section) that our manifold is orientable. We then define the Bopp-Podolsky-Proca Lagrangian $\mathcal{L}_{BPP}$ by

$$\begin{aligned}\mathcal{L}_{BPP}(\varphi,A) &= \frac{1}{8\pi}\left|\frac{\partial A}{\partial t}+\nabla\varphi\right|^2-\frac{1}{8\pi}|\nabla\times A|^2\\ &\quad+\frac{m_1^2}{8\pi}\left(|\varphi|^2-|A|^2\right)+\frac{a^2}{8\pi}\mathcal{L}_{Add}(\varphi,A)\,,\end{aligned}\tag{3}$$

where $a\in\mathbb{R}^+$, $\nabla\times=\star d$ is the curl operator ($\star$ is the Hodge dual and d the usual differentiation on forms). There,

$$\mathcal{L}_{Add}(\varphi,A)=\left(-\Delta_g\varphi+\nabla.\partial_tA\right)^2-\left|\overline{\Delta}_gA+\partial_t\left(\nabla\varphi+\partial_tA\right)\right|^2$$

and $\overline{\Delta}_g=\nabla\times\nabla\times=\delta d$ is half the Hodge-de Rham Laplacian for 1-forms (δ is the codifferential). The blue part in (3) is the Maxwell part. The red part in (3) is the Proca part. The orange part in (3) is the Bopp-Podolsky part. As already mentioned a is the Bopp-Podolsky parameter (physically to be small).

Both Proca and Bopp-Podolsky are then corrections of the Maxwell theory. As a remark,

$$\|(\varphi, A)\|^2_{\text{Lorenz}} = |\varphi|^2 - |A|^2 \ ,$$

where the LHS is the Lorenz norm. Therefore we are indeed giving a mass m_1 to the field (φ, A) in the red part (the Proca part) of (3). When Proca comes into the story, this means that we are going to be in a massive version of the theory.

Once we have $\mathcal{L}_{NLS}$ and $\mathcal{L}_{BPP}$ we define the total action functional $\mathcal{S}_{tot}$ by

$$\mathcal{S}_{tot} = \int \int (\mathcal{L}_{NLS} + \mathcal{L}_{BPP})\, dv_g dt \ .$$

We assume that ψ is of the form $\psi = ue^{iS}$ (polar form) with $u \geq 0$. Then $\mathcal{S}_{tot}$ becomes a function of the 4 variables u, S, φ and A. Taking the variation of $\mathcal{S}_{tot}$ with respect to u, S, φ, and A, we get four equations which, pulled together, form the full Bopp-Podolsky-Schrödinger-Proca system.

The game here is purely variational. In the process we have to take the derivative of terms involving 1-forms like $A \to \int |\nabla \times A|^2$. This involves basic differential calculus together with elementary Hodge de Rham theory. If we let ω_g be the volume form of (M, g), then

$$\frac{1}{2}\left(\frac{d}{dA}\int |\nabla \times A|^2\right).(B) = \int (\star dA, \star dB)\,\omega_g \qquad (\text{quadratic} + \nabla\times = \star d)$$

$$= (-1)^{n-1}\int (\star dA, (\star d \star) \star B)\,\omega_g \qquad (\star\star = (-1)^{n-1} \text{ in } \Lambda^1)$$

$$= \int (\star dA, \delta \star B)\,\omega_g \qquad (\delta = (-1)^{n-1} \star d \star \text{ in } \Lambda^{n-1})$$

$$= \int (d \star dA, \star B)\,\omega_g \qquad (\text{Stokes formula})$$

$$= \int (\star \delta dA, \star B)\,\omega_g \qquad (d\star = \star\delta \text{ in } \Lambda^2)$$

$$= \int (\star \delta dA) \wedge (\star \star B) \qquad (\text{since } \alpha \wedge (\star\beta) = (\alpha, \beta)\omega_g \text{ in } \Lambda^p)$$

Hence,

$$\frac{1}{2}\left(\frac{d}{dA}\int|\nabla\times A|^2\right).(B)$$
$$=\int(\star\delta dA)\wedge(\star\star B)$$
$$=(-1)^{n-1}\int(\star\delta dA)\wedge B \qquad (\star\star=(-1)^{n-1}\text{ in }\Lambda^1)$$
$$=\int(\delta dA,B)\,\omega_g \qquad (\alpha\wedge\beta=(-1)^{n-1}\beta\wedge\alpha\text{ for }\alpha\in\Lambda^{n-1},\beta\in\Lambda^1)$$

Thus,

$$\frac{1}{2}\left(\frac{d}{dA}\int|\nabla\times A|^2\right).(B)=\int(\overline{\Delta}_g A,B)$$

for all B, where $\overline{\Delta}_g=\delta d$ is half of the Hodge-de Rham Laplacian acting on forms.

We return to

$$\mathcal{S}_{tot} = \int\int \left(\mathcal{L}_{NLS} + \mathcal{L}_{BPP}\right) dv_g dt \ .$$

We let ψ be of the form $\psi = ue^{iS}$ (polar form) with $u \geq 0$. We take the variation of $\mathcal{S}_{tot}$ with respect to u, S, φ, and A, and ask that all these partial contributions should be zero (critical point). Then we get four equations which, pulled together, form the full Bopp-Podolsky-Schrödinger-Proca system :

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \left(\hbar\frac{\partial S}{\partial t} + q\varphi + \frac{\hbar^2}{2m_0^2}|\Psi(A,S)|^2\right) u = u^{p-1} \\ 2u\frac{\partial u}{\partial t} + \frac{\hbar}{m_0^2}\nabla.\left(\Psi(A,S)u^2\right) = 0 \\ -\frac{a^2}{4\pi}\Delta_g M(\varphi,A) - \frac{a^2}{4\pi}\frac{\partial}{\partial t}\nabla.N(\varphi,A) - \frac{1}{4\pi}\nabla.\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + \frac{m_1^2}{4\pi}\varphi = qu^2 \\ \frac{a^2}{4\pi}Q(\varphi,A) + \frac{1}{4\pi}\overline{\Delta}_g A + \frac{1}{4\pi}\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + \frac{m_1^2}{4\pi}A = \frac{\hbar q}{m_0^2}\Psi(A,S)u^2 \ , \end{cases}$$

where $\Psi(A,S) = \nabla S - \frac{q}{\hbar}A$, $M(\varphi,A) = -\Delta_g\varphi + \nabla.\partial_t A$,
$N(\varphi,A) = \overline{\Delta}_g A + \partial_t\left(\nabla\varphi + \partial_t A\right)$,
$Q(\varphi,A) = \overline{\Delta}_g N(\varphi,A) + \frac{\partial^2}{\partial t^2}N(\varphi,A) - \nabla\frac{\partial}{\partial t}M(\varphi,A)$.

Letting $a = 0$ in this system the Bopp-Podolsky contribution disappears and we get the Maxwell-Schrödinger-Proca system. Letting $m_1 = 0$ the Proca contribution disappears. The physics associated to the two parameters a and m_1 are :

Value of a	Value of m_1	The physics behind
$a \neq 0$	$m_1 \neq 0$	Bopp-Podolsky-Schrödinger-Proca
$a = 0$	$m_1 \neq 0$	Maxwell-Schrödinger-Proca
$a \neq 0$	$m_1 = 0$	Bopp-Podolsky-Schrödinger
$a = 0$	$m_1 = 0$	Maxwell-Schrödinger

Schrödinger is for the particle ψ. The others are for the electromagnetic field (A, φ). The Maxwell-Proca model for (A, φ) is sometimes referred to as the De Broglie-Proca model.

NOTE : Maxwell-Schrödinger-Proca, when restricted to solitary waves solutions (e.g. when the system becomes elliptic), takes the name of Schrödinger-Poisson-Proca.

III. What is the physics behind Maxwell-Proca

The Maxwell-Schrödinger-Proca system is the big system we wrote down a few slides ago when $a = 0$. In this section we somehow go backward with respect to what the dispersive community do. Let the electric field E, the magnetic induction H, the charge density ρ, and the current density J be given by

$$E = -\frac{1}{4\pi}\left(\frac{\partial A}{\partial t} + \nabla\varphi\right)\ ,\ H = \frac{1}{4\pi}\nabla \times A\,,$$
$$\rho = qu^2\ ,\ J = \frac{\hbar q}{m_0^2}\left(\nabla S - \frac{q}{\hbar}A\right)u^2\,.$$

The two last equations in the Maxwell-Schrödinger-Proca system are

$$\begin{cases} -\frac{1}{4\pi}\nabla.\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + \frac{m_1^2}{4\pi}\varphi = qu^2\ , \\ \frac{1}{4\pi}\overline{\Delta}_g A + \frac{1}{4\pi}\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + \frac{m_1^2}{4\pi}A = \frac{\hbar q}{m_0^2}\Psi(A,S)u^2\ , \end{cases}$$

and $\delta = -\nabla.$ (codifferential $=$ minus the divergence).

Since $\overline{\Delta}_g = \nabla \times \nabla \times \ (n = 3)$, they rewrite as

$$\nabla . E + \frac{m_1^2}{4\pi}\varphi = \rho \quad \text{and} \quad \nabla \times H - \frac{\partial E}{\partial t} + \frac{m_1^2}{4\pi} A = J \ .$$

In other words the two last equations in the MSP system give rise to the first pair of the Maxwell-Proca equations with respect to a matter distribution whose charge and current density are respectively ρ and J. As usual with Maxwell equations, we get for free that the second pair of the equations holds true, and thus : the two last equations in the MSP system can be rewritten in the form of the massive modified Maxwell-Proca equations in SI units :

$$\nabla . E = \rho/\varepsilon_0 - \mu^2 \varphi \ ,$$

$$\nabla \times H = \mu_0 \left(J + \varepsilon_0 \frac{\partial E}{\partial t} \right) - \mu^2 A \ ,$$

$$\nabla \times E + \frac{\partial H}{\partial t} = 0 \text{ and } \nabla . H = 0 \ ,$$

where $\varepsilon_0 = 1$, $\mu_0 = 1$ and $\mu^2 = \frac{m_1^2}{4\pi}$.

The first equation in the Maxwell-Schrödinger-Proca system is the nonlinear elliptic (time dependent) Schrödinger equation. The second equation in (MSP) is the charge continuity equation $\frac{\partial \rho}{\partial t} + \nabla . J = 0$, which is equivalent to the Lorenz condition

$$\nabla . A + \frac{\partial \varphi}{\partial t} = 0$$

when $m_1 \neq 0$ (and thus as soon as there is a nonzero Proca mass). Indeed, taking the derivation of the first Maxwell equation $\nabla . E = \rho - \mu^2 \varphi$ with respect to time, and the divergence of the second equation $\nabla \times H - \frac{\partial E}{\partial t} = J - \mu^2 A$, we get that

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla . J &= \nabla . \frac{\partial E}{\partial t} + \mu^2 \frac{\partial \varphi}{\partial t} + \nabla . (\nabla \times H) - \nabla . \frac{\partial E}{\partial t} + \mu^2 \nabla . A \\ &= \mu^2 \left(\nabla . A + \frac{\partial \varphi}{\partial t} \right) \end{aligned}$$

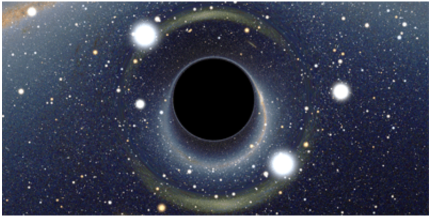
since $\nabla . (\nabla \times H) = \delta(\star d) H$, $\delta = \star^{-1} d \star$ in Λ^1, $\star\star = 1$ in Λ^2, and $d^2 = 0$ so that $\nabla . (\nabla \times H) = 0$. The condition $m_1 \neq 0$ (thus $\mu \neq 0$) breaks the gauge invariance and enforces the Lorenz gauge.

As we just saw, the Maxwell equations in Proca form (the two last equations in our system when $a = 0$) are

$$
\begin{aligned}
&\nabla . E = \rho - \mu^2 \varphi \ , \\
&\nabla \times H - \frac{\partial E}{\partial t} = J - \mu^2 A \ , \\
&\nabla \times E + \frac{\partial H}{\partial t} = 0 \ , \\
&\nabla . H = 0 \ .
\end{aligned}
$$

They reduce to the Maxwell equations as $\mu \to 0$. Proca (1936) was using the Lorenz formalism. Under this form, referred to as the "modern format", the equations appeared for the first time in a paper by Schrödinger : "*The earth's and the sun's permanent magnetic fields in the unitary field theory*" (1943). These equations have been discussed by several physicists. In addition to Proca and Schrödinger : De Broglie, Pauli, Yukawa, Stueckleberg... The point in these theories is that m_1 is nothing but the mass of the photon.

We are talking about a theory where photons have a mass. Recall : "the photon is the quantum of the electromagnetic field including electromagnetic radiation such as light, and the force carrier for the electromagnetic force" (Wikipedia).



The people involved in this early story of massive photons :

VIII.

THE EARTH'S AND THE SUN'S PERMANENT MAGNETIC FIELDS IN THE UNITARY FIELD THEORY.

(From the Dublin Institute for Advanced Studies.)

By ERWIN SCHRÖDINGER.

[Read 28 June. Published 29 November, 1943.]

§ 1. Survey.

For not excessively strong electromagnetic fields in empty space and neglecting gravitation the Unitary Field Theory[1] gives the equations ($c = 1$)

$$\begin{aligned} H &= \operatorname{curl} A \\ E &= -\dot{A} - \operatorname{grad} V \\ \operatorname{curl} H - \dot{E} &= -\mu^2 A \\ \operatorname{div} E &= -\mu^2 V \end{aligned} \qquad (1)$$

and suggests that the constant μ^{-1} be *not* cosmically large (in which case the equations boil down to Maxwell's) but very roughly speaking of the order of the radius of the earth.

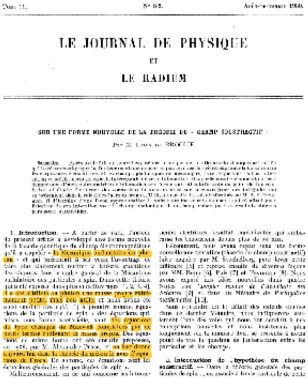
Erwin Schrödinger
1887 – 1961

The Earth's and the Sun's Permanent Magnetic Fields in the Unitary Field Theory

Erwin Schrödinger

Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences

Vol. 49 (1943/1944), pp. 135-148



LE JOURNAL DE PHYSIQUE

ET

LE RADIUM

SUR UNE FORME NOUVELLE DE LA THÉORIE DU « CHAMP SOUSTRACTIF »

Louis de Broglie
1892 – 1987

Sur une forme nouvelle de la théorie du « champ soustractif »

Louis de Broglie

J. Phys. Radium, 1950, 11 (8-9), pp.481-489.

"Starting 1934, the author of the present paper developed a new form of the quantum theory of electromagnetic filed that he called "la mécanique ondulatoire du photon" and which, according to him, had the advantage of a clear insertion of quantum field theory in the general framework of wave mechanics of spin particles. In this theory, which has been exposed in different papers, it has been given to the photon a specific mass, extremely small, but nonzero, and, therefore, we were led starting 1934 to take for the equations for spin 1 particles equations which, in vectorial form, are classical Maxwell equations completed with small terms involving the specific mass. Similar equations have then been proposed, in 1936, by Mr Alexandru Proca, and they are now given the name of Proca equations in the theory of mesons."

IV. The physics behind Bopp-Podolsky-Proca

We briefly discuss the Bopp-Podolsky-Proca equations in vaccum.

Take the two last equations in the Maxwell-Schrödinger-Proca system with the unit $c = 1$ and cancel the u-terms in these equations (take $u = 0$). Then you get the equations for the Maxwell-Proca electrodynamics with Lorenz condition. Namely,

$$\Box\Phi + m_1^2\Phi = 0 \ , \tag{4}$$

where Φ represents the full field consisting of φ and A, and $\Box = \frac{\partial^2}{\partial t^2} + \Delta_g$ is the d'Alembert operator. The Lorenz condition (cf. before) gives that $\partial_t\varphi = \delta A$.

Equation (4) describes photons with (small) mass m_1.

The equations for the Bopp-Podolsky electrodynamics in vacuum are

$$a^2\Box^2\Phi + \Box\Phi = 0 \; . \tag{5}$$

The equations in the case of Bopp-Podolsky-Proca are

$$a^2\Box^2\Phi + \Box\Phi + m^2\Phi = 0 \; . \tag{6}$$

The traditional interpretation for (5) is that the equation splits into two second order equations

$$\begin{aligned} \Box\hat{\Phi} &= 0 \; , \\ \Box\tilde{\Phi} + \frac{1}{a^2}\tilde{\Phi} &= 0 \; , \end{aligned} \tag{7}$$

where $\hat{\Phi} = a^2\Box\Phi + \Phi$ and $\tilde{\Phi} = a^2\Box\Phi$. These two equations give two kinds of photons. The first equation in (7) describes massless photons and the second equation in (7) describes massive photons (with mass of the order of $1/a$). They are sometimes referred to as Bopp-Podolsky photons, Podolsky dark photons or just dark photons. A theory with massless and massive photons requires fourth order equations.

A somehow similar interpretation can be given for (6). Define

$$\begin{cases} \hat{\Phi} = \Box\Phi + \frac{1+\sqrt{\Delta}}{2a^2}\Phi \\ \tilde{\Phi} = \Box\Phi + \frac{1-\sqrt{\Delta}}{2a^2}\Phi \ , \end{cases}$$

where $\Delta = 1 - 4a^2m^2$. Then

$$\begin{cases} \Box\hat{\Phi} + \frac{1-\sqrt{\Delta}}{2a^2}\hat{\Phi} = 0 \\ \Box\tilde{\Phi} + \frac{1+\sqrt{\Delta}}{2a^2}\tilde{\Phi} = 0 \ . \end{cases} \tag{8}$$

In this situation, we recover photons with "small" mass of the order of m by the first equation in (8), and massive photons with mass of the order of $1/a$ by the second equation in (8), the point here being that

$$\frac{1-\sqrt{\Delta}}{2a^2} \simeq m^2 \ \text{ and } \ \frac{1+\sqrt{\Delta}}{2a^2} \simeq \frac{1}{a^2}$$

as $a \to 0^+$.

V. Why we do need Proca in the closed setting

It is somehow odd to add Proca to Bopp-Podolsky. One of the primary objective of Bopp-podolsky was to recover gauge invariance, and we break this again by adding the Proca mass. An important point here is that one must add Proca in the context of closed manifolds if we want to say something nontrivial about the equations.

The static case we are going to discuss (where $\partial_t A = 0$, $\partial_t \varphi = 0$, $\partial_t u = 0$) makes that the two last equations in the full Bopp-Podolsky-Schrödinger-Proca system (and thus also in the full Maxwell-Schrödinger-Proca system which corresponds to $a = 0$) are

$$\begin{cases} a^2\Delta_g^2\varphi + \Delta_g\varphi + m_1^2\varphi = 4\pi q u^2 \ , \\ a^2\Delta_g^2 A + \Delta_g A + m_1^2 A = \frac{4\pi\hbar q}{m_0^2}\Psi(A,S)u^2 \ , \end{cases} \tag{9}$$

where $\Psi(A,S) = \nabla S - \frac{q}{\hbar}A$. Then :

If (u, φ, A) is a solution of the Proca-free version of (9) *(where $m_1 = 0$), namely of*

$$\begin{cases} a^2\Delta_g^2\varphi + \Delta_g\varphi = 4\pi q u^2 \ , \\ a^2\Delta_g^2 A + \Delta_g A = \frac{4\pi\hbar q}{m_0^2}\Psi(A,S)u^2 \ , \end{cases}$$

where $\Psi(A,S) = \nabla S - \frac{q}{\hbar}A$, then $u = 0$, φ is constant and also, A is zero when the Ricci curvature of the manifold is positive.

This is very much a consequence of the fact that M is closed. Integrating the first equation gives that $u \equiv 0$. Then, if we multiply the same equation by $\Delta_g\varphi$ and integrate over M, we get that $\Delta_g\varphi \equiv 0$, and thus that φ is a constant. The second equation gives that $\Delta_g A = 0$. Since there are no nontrivial harmonic 1-forms when the Ricci curvature is positive (an immediate consequence of the Bochner-Lichnerowicz-Weitzenböck formula), we get that $A \equiv 0$ needs to be trivial.

Part 2

The math part

VI. Reduction of the equations

We want to pass from this

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \left(\hbar\frac{\partial S}{\partial t} + q\varphi + \frac{\hbar^2}{2m_0^2}|\Psi(A,S)|^2\right)u = u^{p-1} \\ 2u\frac{\partial u}{\partial t} + \frac{\hbar}{m_0^2}\nabla.\left(\Psi(A,S)u^2\right) = 0 \\ -\frac{1}{4\pi}\nabla.\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) - \frac{a^2}{4\pi}\Delta_g M(\varphi,A) - \frac{a^2}{4\pi}\frac{\partial}{\partial t}\nabla.N(\varphi,A) + \frac{m_1^2}{4\pi}\varphi = qu^2 \\ \frac{1}{4\pi}\overline{\Delta}_g A + \frac{1}{4\pi}\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + \frac{m_1^2}{4\pi}A + \frac{a^2}{4\pi}Q(\varphi,A) = \frac{\hbar q}{m_0^2}\Psi(A,S)u^2 , \end{cases}$$

to this

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \Phi(x,v,A)u = u^{p-1} \\ a^2\Delta_g^2 v + \Delta_g v + m_1^2 v = 4\pi q u^2 \\ a^2\Delta_g^2 A + \Delta_g A + m_1^2 A = \frac{4\pi q\hbar}{m_0^2}\Psi(A,S)u^2 . \end{cases} \qquad (BPSP)_a$$

In order to do this we assume that we are in the static case of the system, and therefore that $\partial_t\varphi \equiv 0$ and $\partial_t A \equiv 0$. Then we look for solitary wave type solutions like ue^{iS} with $\partial_t u \equiv 0$ and $S(x,t) = S(x) + \frac{\omega^2}{\hbar}t$.

Such type of solutions (when $\partial_t u \equiv 0$, $\partial_t \varphi \equiv 0$, $\partial_t A \equiv 0$ and $S(x,t) = S(x) + \frac{\omega^2}{\hbar} t$) are referred to as electro-magneto-static solutions. They were considered by Benci and Fortunato for the Klein-Gordon-Maxwell equations in $\mathbb{R}^3$. We get standing waves solutions when $S(x) \equiv 0$. Under these assumptions, the full Bopp-Podolosky-Schrödinger-Proca system becomes

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \Phi(x,\varphi,A)u = u^{p-1} \\ \nabla.\left(\Psi(A,S)u^2\right) = 0 \\ a^2\Delta_g^2\varphi + \Delta_g\varphi + m_1^2\varphi = 4\pi q u^2 \\ a^2\overline{\Delta}_g^2 A + \overline{\Delta}_g A + m_1^2 A = \frac{4\pi\hbar q}{m_0^2}\Psi(A,S)u^2 \,. \end{cases} \tag{10}$$

By the fourth equation in (10), since $\nabla.\overline{\Delta}_g = 0$ (as $\delta^2 = 0$), the second equation in (10) is nothing but the Coulomb gauge condition $\delta A = 0$, and the three other equations give rise to $(BPSP)_a$ by noting that when $\delta A = 0$ we get that $\overline{\Delta}_g A = \Delta_g A$, where $\Delta_g = d\delta + \delta d$ is the Hodge-de Rham Laplacian on forms.

$$(10) = (BPSP)_a + \text{“}\delta A = 0\text{''} \,.$$

VII. One result on these equations

Summarizing, the background space is a 3-dimensional closed Riemannian manifold (M, g). The BPSP system we discuss is

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \Phi(x, v, A)u = u^{p-1} \\ a^2\Delta_g^2 v + \Delta_g v + m_1^2 v = 4\pi q u^2 \\ a^2\Delta_g^2 A + \Delta_g A + m_1^2 A = \frac{4\pi q\hbar}{m_0^2}\Psi(A, S)u^2 \ . \end{cases} \qquad (BPSP)_a$$

The unknowns are (u, v, A), where u and v are functions, $u \geq 0$ in M, and A is a 1-form (and $v = \varphi$). The whole system corresponds to an electro-magneto-static regime. Here :

$$\Phi(x, v, A) = \frac{\hbar^2}{2m_0^2}|\Psi(A, S)|^2 + \omega^2 + qv \ ,$$

and $\Psi(A, S) = \nabla S - \frac{q}{\hbar}A$. Then $q, m_0, m_1 > 0$ and $\omega \in \mathbb{R}$. The Bopp-Podolsky parameter is $a \geq 0$, $\hbar$ is the reduced Planck's constant and $p \in (2, 6]$.

Letting $a = 0$ in $(BPSP)_a$ we get the SPP system

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \Phi(x,v,A)u = u^{p-1} \\ \Delta_g v + m_1^2 v = 4\pi q u^2 \\ \Delta_g A + m_1^2 A = \frac{4\pi q\hbar}{m_0^2}\Psi(A,S)u^2 . \end{cases} \qquad (SPP)$$

Here again the unkowns are $(u, v, A) =$ (function,function,1-form) and we require that $u \geq 0$. In particular

$$(SPP) = (BPSP)_a \text{ when } a = 0 .$$

As a remark : $p = 6$ is the critical Sobolev exponent. The two systems become critical from the viewpoint of Sobolev embeddings in that case (e.g. like the Yamabe equation was critical).

QUESTION : In which sense can we write that

$$\lim_{a\to 0^+} (BPSP)_a = (SPP) \quad ?$$

Concretly, can we avoid the formation of blowing-up solutions as $a \to 0^+$? Critical equations tend to create blowing-up solutions.

Theorem (H., 2021)

Let (M,g) be a smooth closed 3-manifold, $\omega \in \mathbb{R}$, $a, q, m_0, m_1 > 0$ be positive real numbers and $S \in C^\infty_R(M)$ be a smooth real-valued function. Let $p \in [\frac{22}{5}, 6]$. We assume that $Rc_g + m_1^2 g > 0$ in the sense of bilinear forms and when p is critical from the viewpoint of Sobolev embeddings, namely when $p = 6$, we also assume that

$$\omega^2 + \frac{\hbar^2}{2m_0^2}|\nabla S|^2 < \frac{\hbar^2}{2m_0^2}\Lambda_g \qquad (CritCond)$$

in M, where $\Lambda_g > 0$ is smooth and such that $\Delta_g + \Lambda_g$ has nonnegative mass. Then

$$\lim_{a \to 0^+} (BPSP)_a = (SPP)$$

strongly in $C^{2,\theta}_R \times C^{0,\theta}_R \times C^{0,\theta}_V$ for some $\theta \in (0,1)$. The result remains valid if we add to the equations the Lorenz condition.

In other words we prove that as the Bopp-Podolsky parameter a goes to zero, any family of solutions (whatever the family of solutions we consider) to the Bopp-Podolsky-Schrödinger-Proca systems strongly converges (i.e. with a very good/strong control on the convergence) to a solution of the Schrödinger-Poisson-Proca system. Then a fourth order system strongly converges (without collapsing and blow-up) to a second order system.

Concretely, we let $(a_\alpha)_\alpha$ be any sequence of positive real numbers such that $a_\alpha \to 0$ as $\alpha \to +\infty$. We let $(m_\alpha)_\alpha$ be any sequence of positive real numbers such that $m_\alpha \to m_1$ as $\alpha \to +\infty$. Let $(BPSP)_\alpha$ be the BPSP-system with a_α and m_α. Then, what the theorem says is that (under the assumptions of the theorem) for any sequence $((u_\alpha, v_\alpha, A_\alpha))_\alpha$ of solutions of $(BPSP)_\alpha$, there holds that, up to passing to a subsequence, $(u_\alpha, v_\alpha, A_\alpha) \to (u, v, A)$ in $C_R^{2,\theta} \times C_R^{0,\theta} \times C_V^{0,\theta}$ as $\alpha \to +\infty$, where (u, v, A) is a solution of (SPP) . Improvements are that the convergence actually holds in $C_R^{2,\theta} \times H_R^2 \times H_V^2$, that $u > 0$, $v > 0$ as soon as $(u_\alpha)_\alpha$ is nontrivial, and that $A \not\equiv 0$ as soon as $(u_\alpha)_\alpha$ is nontrivial and $\nabla S \not\equiv 0$.

The subscripts R and V in space notations are for real valued (R for real valued functions) and vector valued (V for vector valued, in our context 1-forms).

A few words on the positive mass theorem of Schoen-Yau and Witten which we use in the critical case of our equations are as follows. Let Λ_g be a smooth function in M. An operator like $\Delta_g + \Lambda_g$ is said to be coercive if its energy controls the square of the H^1-norm. Then a coercive operator like $\Delta_g + \Lambda_g$ is said to have positive mass (resp. nonnegative mass) if the regular part of its Green's function is positive (resp. nonnegative) on the diagonal of $M \times M$.

By the positive mass theorem of Schoen-Yau and Witten, there exists a function Λ_g (with $\Lambda_g > 0$ in M) such that $\Delta_g + \Lambda_g$ has positive mass on any 3-manifold whose scalar curvature is positive. In the case of the standard 3-sphere we can take $\Lambda_g = \frac{3}{4}$, and this is the best possible value. The condition (CritCond) is a stationary wave Schoen's type condition.

VIII. Few words on the proof of our theorem

The proof goes in several steps of increasing difficulties. In what follows $a > 0$, $m_a > 0$ and the m_a's are assumed to be such that $m_a \to m_1$ as $a \to 0$, where $m_1 > 0$.

Step 1 : Interdependency.

For any $u \in H^1_R$, there exists a unique $A_a(u) \in H^4_V \cap C^2_V$ such that

$$a^2\Delta_g^2 A_a(u) + \Delta_g A_a(u) + \left(m_1^2 + \frac{4\pi q^2}{m_0^2}u^2\right) A_a(u) = \frac{4\pi q\hbar}{m_0^2}(\nabla S)u^2 \ ,$$

and there exists a unique $v_a(u) \in H^4_R \cap C^2_R$ such that

$$a^2\Delta_g^2 v_a(u) + \Delta_g v_a(u) + m_a^2 v_a(u) = 4\pi q u^2 \ .$$

The proof goes through basic variational arguments. We are dealing here with equations 2 and 3 in the BPSP-system.

Now we want to get estimates on $A_a(u)$ and $v_a(u)$ which do not depend on a as $a \to 0^+$. Dealing with $A_a(u)$ is more difficult. A preliminary estimate we get from the the equation for $A_a(u)$ and the Bochner-Lichnerowicz-Weitzenböck formula is the following.

Step 2 : Preliminary estimates on $A_a(u)$.

There exists an explicit constant $C > 0$, depending only on g, S and m_0, such that $\|A_a(u)\|_{H^1_V} \leq C\|u\|_{L^2_R}$ for all $u \in H^1_R$ and all $0 < a \ll 1$.

This we can get by contracting by $A_a(u)$ the equation for $A_a(u)$, then integrating, using the BLW-formula

$$(\Delta_g A, A) = \frac{1}{2}\Delta_g |A|^2 + |\nabla A|^2 + \mathrm{Rc}_g(A^\sharp, A^\sharp) \ ,$$

using that $Rc_g + m_1^2 g > 0$ and then throwing away the a-terms (using their sign). $\square$

In order to go on we need a little control on the u's in the system. An easy be key starting step is as follows.

Step 3 : Starting with low estimates on u.

There exists $C > 0$, independent of a, such that for any solution $u \in H_R^1$, $u \geq 0$, of the first equation in $(BPSP)_a$, and any $0 < a \ll 1$, $\|u\|_{L_R^{p-2}} \leq C$.

For instance, in the critical case $p = 6$ we control u in L_R^4. By the maximum principle, either $u \equiv 0$ or $u > 0$ in M. By our sign convention on Δ_g, integrating by parts, and since M has no boundary, there holds that

$$\int_M \frac{\Delta_g u}{u} dv_g \leq 0 .$$

The estimate follows by multiplying the first equation in $(BPSP)_a$ by $1/u$ and integrating over M. In the process we use Step 2. $\square$

By the above and the Gidas-Spruck subcritical blow-up argument we can settle the subcritical case in the equations. This is where we use $p \geq \frac{22}{5}$.

Step 4 : The Gidas-Spruck argument.

Suppose $\frac{22}{5} \leq p < 6$. *There exists* $C > 0$, *independent of* a, *such that for any solution* $u \in H^1_R$, $u \geq 0$, *of the first equation in* $(BPSP)_a$, *and any* $0 < a \ll 1$, $\|u\|_{L^\infty_R} \leq C$.

By Hölder's inequality, and the Sobolev inequality,

$$\int_M u_\alpha^2 |v_\alpha| dv_g \leq \left(\int_M u_\alpha^{12/5} dv_g\right)^{5/6} \left(\int_M |v_\alpha|^6 dv_g\right)^{1/6} \leq C \|u_\alpha\|^2_{L_R^{12/5}} \|v_\alpha\|_{H^1_R} ,$$

where $C > 0$ is independent of α. By Step 3, and since $p \geq \frac{22}{5}$ so that $p - 2 \geq \frac{12}{5}$, we have that $\|u_\alpha\|_{L_R^{12/5}} = O(1)$. By the second equation in $(BPSP)_a$ we the get that $\|v_a(u)\|_{H^1_R} = O(1)$. Together with the control on $A_a(u)$ given by Step 2 we can then branch on the Gidas-Spruck argument. $\square$

The controls we have on $A_a(u)$ and $v_a(u)$ are not sufficient in the critical case $p = 6$. We need there to be more subtle. A key point is to prove the following.

Step 5 : More estimates on $A_a(u)$.

For any $K > 0$, there exists $C_K > 0$ such that $\|A_a(u)\|_{L^\infty_V} \leq C_K$ for all $u \in H^1_R$ such that $\|u\|_{L^4_R} \leq K$, and all $0 < a \ll 1$.

We define a new variable $\Psi = a^2 \Delta_g A + A$. Then $\Delta_g \Psi = \Theta$, where $\Theta = \frac{4\pi q \hbar}{m_0^2} (\nabla S) u^2 - m_a^2 A - \frac{4\pi q^2}{m_0^2} A u^2$. We prove that $\|\Psi\|_{L^q_V} \leq C_q(K)$ for all $q \geq 1$. Then that $\|A\|_{L^q_V} \leq C_q(K)$ for all $q \geq 1$. At last that $\|\Psi\|_{L^\infty_V} \leq C_K$. $\square$

The similar estimates for $v_a(u)$ are much easier to get. The equation for $v_a(u)$ is less intricate then the one for $A_a(u)$ as we do not have a u^2v-term to handle (plus the fact that it is easier to handle functions than 1-forms).

Step 6 : More estimates on $v_a(u)$.

For any $K > 0$, there exists $C_K > 0$ such that $\|v_a(u)\|_{L^\infty_R} \leq C_K$ for all $u \in H^1_R$ such that $\|u\|_{L^4_R} \leq K$, and all $0 < a \ll 1$.

The estimate follows from easy manipulations of the equation satisfied by $v_a(u)$. We actually get a bound in H^2_R which, by Sobolev, implies the L^∞_R-bound. $\square$

Once we have these estimates (plus some complementary others that we do not discuss here), we can branch on advanced blow-up theory.

Step 7 : Advanced blow-up.

Let $(a_\alpha)_\alpha$ be any sequence of positive real numbers such that $a_\alpha \to 0$ as $\alpha \to +\infty$, $(m_\alpha)_\alpha$ be any sequence of positive real numbers such that $m_\alpha \to m_1$ as $\alpha \to +\infty$ and $((u_\alpha, v_\alpha, A_\alpha))_\alpha$ be any sequence of solutions of $(BPSP)_\alpha$. Up to passing to a subsequence, $(u_\alpha, v_\alpha, A_\alpha) \to (u, v, A)$ in $C_R^{2,\theta} \times H_R^2 \times H_V^2$ as $\alpha \to +\infty$, where (u, v, A) is a solution of (SPP).

The blow-up theory we use here is specific to dimension 3 and goes back to the seminal work of Schoen (1988), Li-Zhu (1999), Marques (2005), Druet (2003), Druet-Hebey-Robert (2004). We use it here in the form developed by Hebey and Thizy (2018). Three key points are involved in this approach. We briefly discuss this below.

We proceed as follows. To make things easy, the whole goal is to prove that

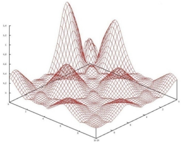
$$\|u_\alpha\|_{L^\infty_R} = O(1)\ . \qquad (\star)$$

When $p < 6$ this has been proved in Step 4 with the Gidas-Spruck argument. Suppose now that $p = 6$. By Step 3 there exists $K > 0$ such that $\|u_\alpha\|_{L^4_R} \le K$. By Steps 5 and 6 we then get that $\|A_\alpha\|_{L^\infty_R} = O(1)$ and $\|v_\alpha\|_{L^\infty_R} = O(1)$. In particular, if we let $\Phi_\alpha = \Phi(\cdot, v_\alpha, A_\alpha)$, then

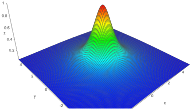
$$\|\Phi_\alpha\|_{L^\infty_R} = O(1)\ .$$

With such an estimate we can apply the first key point of the 3-dimensional blow-up analysis (Schoen, Li-Zhu) which asserts that bubbles need to be isolated. We then get that $\|u_\alpha\|_{H^1_R} = O(1)$.

Typically we could have had cluster's type configurations for the u_α's (groups of bubbles interacting one with another) such as



But it turns out that in dimension 3, when the potential in "Yamabe" type equations is controlled in L^∞ (this is where we use Steps 5 and 6), we actually always do have configurations with a repetition of single bumps like



This implies that we have a finite number of such bumps, then that the u_α's are bounded in H^1_R (and thus also in L^6_R). With a slight modification of Steps 5 and 6 we get that the A_α's and v_α's are bounded in H^2-spaces (as the u_α's are bounded in L^6_R).

Suppose now by contradiction that for some subsequence of the u_α's the L^∞-norm of the u_α's goes to $+\infty$. Up to passing to a subsequence again, because of the H^2 bounds on A and v, we can assume that $A_\alpha \to A$ in $C_V^{0,\theta}$, $v_\alpha \to v$ in $C_R^{0,\theta}$ and $\Phi_\alpha \to \Phi$ in $C_R^{0,\theta}$ for some $0 < \theta < 1$, where $\Phi = \Phi(\cdot, v, A)$. By the H^1-estimate on the u_α's we may also assume that $u_\alpha \rightharpoonup u$ in H_R^1 and that $u_\alpha \to u$ in L_R^4 as $\alpha \to +\infty$. A second key point in the 3-dimensional blow-up analysis (which goes back to Druet) gives that $u \equiv 0$. Then, by Step 2, $A \equiv 0$ and it is easily seen that we also need to have that $v \equiv 0$. Then

$$\Phi = \omega^2 + \frac{\hbar^2}{2m_0^2}|\nabla S|^2 .$$

A third and last key point of the 3-dimensional blow-up analysis (Druet-Hebey-Robert, Li-Zhu, Marques, Schoen) asserts that when blow-up occurs the limit operator cannot have positive mass. Here, by (CritCond) and the maximum principle, the limit operator $\frac{\hbar^2}{2m_0^2}\Delta_g + \Phi$ has positive mass. A contradiction. $\square$

Thank you for your attention