Sharp Sobolev-Poincaré inequalities on compact Riemannian manifolds

by

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December 2001
Augmented May 2002

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Introduction

Given (M,g) a smooth compact n-dimensional Riemannian manifold, one easily defines the Sobolev spaces $H_k^p(M)$, following what is done in the more traditionnal Euclidean context. For instance, when $k = 1$, and $p = 2$, one may define the Sobolev space $H_1^2(M)$ as follows: for $u \in C^\infty(M)$, we let

$$\|u\|_{H_1^2}^2 = \|u\|_2^2 + \|\nabla u\|_2^2$$

where $\|.\|_p$ is the L^p-norm with respect to the Riemannian measure dv_g. We then define $H_1^2(M)$ as the completion of $C^\infty(M)$ with respect to $\|.\|_{H_1^2}$. Very useful properties of H_1^2 (more generally of H_1^p, $p \geq 1$) are that Lipschitz functions on M do belong to the Sobolev space $H_1^2(M)$, and that if $u \in H_1^2(M)$, then $|u| \in H_1^2(M)$ and $|\nabla |u|| = |\nabla u|$ almost everywhere.

As for bounded open subsets of the Euclidean space, the Sobolev embedding theorem (continuous embeddings), and the Rellich-Kondrakov theorem (compact embeddings), do hold. Assume that $n \geq 3$, and let $2^\star = \frac{2n}{n-2}$ be the critical Sobolev exponent. Then for any $p \in [1, 2^\star]$, $H_1^2(M) \subset L^p(M)$ and this embedding is continuous, with the property that it is also compact if $p < 2^\star$. The Sobolev inequality corresponding to the critical continuous embedding $H_1^2(M) \subset L^{2^\star}(M)$ can be written as follows: for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \leq A_1 \|\nabla u\|_2^2 + B_1 \|u\|_2^2 \tag{0.1}$$

where A_1 and B_1 are positive constants independent of u, but that may depend on the manifold. Another very useful inequality, closely related to the Sobolev inequality, is the so-called Poincaré inequality. In the particular case of the H_1^2-Sobolev space, the Poincaré inequality reduces to the Rayleigh characterization of the first nonzero eigenvalue of the Laplacian: there exists a positive constant A_2 such that for any $u \in H_1^2(M)$,

$$\|u - \overline{u}\|_2^2 \leq A_2 \|\nabla u\|_2^2 \tag{0.2}$$

where $\overline{u} = V_g^{-1} \int_M u dv_g$ is the average of u, and V_g the volume of M with respect to g. In particular, it easily follows from (0.2) that for any $u \in H_1^2(M)$,

$$\|u\|_2^2 \leq A_3 \|\nabla u\|_2^2 + B_3 \|u\|_1^2 \tag{0.3}$$

where A_3 and B_3 are positive constants independent of u, but that may depend on the manifold. Such an inequality appeared first in the Courant and Hilbert monograph [9]. Combining (0.1) and (0.3), we do get that there exist positive constants A and B such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \leq A \|\nabla u\|_2^2 + B \|u\|_1^2 \tag{0.4}$$

This inequality was considered in Nirenberg [31]. We refer to this inequality as the Sobolev-Poincaré inequality.

These notes are devoted to the study of the sharp form of (0.4) with respect to the first constant. They are both a combination of a series of three papers by Druet-Hebey [18], Druet-Hebey-Vaugon [19] and Hebey [25], and an expended version of a series of lectures given by the author at various places like the university of Texas at Austin, Princeton university, the university of British Columbia, and the scuola normale superiore di Pisa. New results are also presented. The author wishes to express his gratitude to the above institutions for their warm hospitality.

1 Few words on the Euclidean space

It is known since the work of Sobolev [37] that there exists a positive constant K such that for any $u \in C_0^\infty(I\!R^n)$, the space of smooth functions with compact support in $I\!R^n$,

$$\|u\|_{2^\star}^2 \le K\|\nabla u\|_2^2 \tag{1.1}$$

More direct arguments were later on discovered in independent works by Gagliardo [21] and Nirenberg [31]. These different approaches of Gagliardo, Nirenberg, and Sobolev do not give the value of the best constant K in (1.1). A discussion of the sharp form of (1.1) restricted to the case $n = 3$ appeared first in Rosen [35]. Then we find independent works by Aubin [3] and Talenti [38] where the sharp form of (1.1) is given. If K_n stands for the best constant in (1.1), it was shown by these authors that

$$K_n = \frac{4}{n(n-2)\omega_n^{2/n}}$$

where ω_n is the volume of the unit n-sphere. The sharp Sobolev inequality then reads as

$$\|u\|_{2^\star}^2 \le K_n\|\nabla u\|_2^2 \tag{1.2}$$

and it is easily seen that equality holds in (1.2) if u has the form

$$u = \left(\lambda + |x - x_0|^2\right)^{1-\frac{n}{2}} \tag{1.3}$$

where λ is any positive constant and x_0 is any point in $I\!R^n$. Both the approaches in [3] and [38] were based on previous work by Bliss [5] where K_n was computed for radially symmetric functions. By standard Morse theory, it suffices to prove (1.2) for continuous nonnegative functions u with compact support $\overline{\Omega}$, $\overline{\Omega}$ being itself smooth, u being smooth in $\overline{\Omega}$ and such that it has only nondegenerate critical points in $\overline{\Omega}$. For such an u, let $u^\star : I\!R^n \to I\!R$, radially symmetric, nonnegative, and decreasing in $|x|$ be defined by

$$Vol_\delta\left(\{x \in I\!R^n, u^\star(x) \ge t\}\right) = Vol_\delta\left(\{x \in I\!R^n, u(x) \ge t\}\right)$$

where δ stands for the Euclidean metric, and $Vol_\delta X$ for the Euclidean volume of X. It is easily seen that $u^\star$ has compact support and is Lipschitz. Moreover, the co-area formula gives that for any $m \ge 1$,

$$\int_{I\!R^n} |\nabla u|^m dx \ge \int_{I\!R^n} |\nabla u^\star|^m dx$$

and

$$\int_{I\!R^n} |u|^m dx = \int_{I\!R^n} |u^\star|^m dx$$

It follows that it suffices to prove (1.2) for decreasing absolutely continuous radially symmetric functions which equal zero at infinity, and we are back to the Bliss argument.

2 Sharp Sobolev-Poincaré inequalities - The questions

Mimicking what has been done for the standard Sobolev inequality, see Hebey [24] and Druet-Hebey [17] for expositions in book form, the goal in these notes is to discuss the sharp form of (0.4) with respect to its first constant. Given (M, g) smooth, compact, of dimension $n \geq 3$, we define the sharp constant $A_s(M)$ in (0.4) by

$$A_s(M) = \inf \left\{ A \text{ s.t. } \exists B \text{ for which (0.4) holds with } A \text{ and } B \right\}$$

where, by saying that (0.4) holds with A and B, we mean that (0.4) holds with A and B for all functions $u \in H_1^2(M)$. The first question to consider is whether or not we can compute the value of $A_s(M)$. It turns out that the answer to this question is simple and follows from local comparison arguments with the Euclidean metric. More precisely, we will return to these statements in section 5, it is easily seen that the two following propositions hold:

(1) any constant A in (0.4), whatever B and the manifold (M, g) are, has to be such that $A \geq K_n$, and

(2) for any (M, g), and any $\varepsilon > 0$, there exists $B_\varepsilon > 0$ such that (0.4) holds with $A = K_n + \varepsilon$ and $B = B_\varepsilon$.

In other words, (2) says that for any smooth compact Riemannian manifold (M, g) of dimension $n \geq 3$, and for any $\varepsilon > 0$, there exists a positive constant B_ε such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \leq (K_n + \varepsilon) \|\nabla u\|_2^2 + B_\varepsilon \|u\|_1^2 \tag{2.1}$$

It clearly follows from (1) that $A_s(M) \geq K_n$. It clearly follows from (2) that $A_s(M) \leq K_n + \varepsilon$ for all $\varepsilon > 0$. Hence, (1) and (2) give that for any smooth compact Riemannian manifold (M, g) of dimension $n \geq 3$, $A_s(M) = K_n$. In particular, $A_s(M)$ does not depend on the manifold. This is not anymore the case for B_ε. Taking $u \equiv 1$ in (2.1), it is easily seen that $B_\varepsilon \geq V_g^{-(n+2)/n}$ where V_g is the volume of M with respect to g. In particular, B_ε has to depend on the manifold.

Now we consider what we refer to as the sharp Sobolev-Poincaré inequality. Given (M, g) a smooth compact Riemannian manifold of dimension $n \geq 3$, we say that the sharp Sobolev-Poincaré inequality is true on (M, g) if there exists a positive constant B such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \leq K_n \|\nabla u\|_2^2 + B \|u\|_1^2 \tag{2.2}$$

Depending on the manifold, (2.2) may be true, or may not be true. The first question we ask is the following:

Question 1: *Given a smooth compact Riemannian manifold (M, g) of dimension $n \geq 3$, is (2.2) true on (M, g) ?*

If this is the case, a similar statement is that $A_s(M)$ is attained in (0.4), or also that we can take $\varepsilon = 0$ in (2.1). Now we distinguish two cases, depending on whether question 1 receives a positive answer or not.

In the first case, we assume that the manifold we consider is such that (2.2) is true. Then we can saturate (2.2) with respect to the remaining constant B. More precisely, when (2.2) is true, we define $B_0(g)$ by

$$B_0(g) = \inf \left\{ B \text{ s.t. (2.2) is true} \right\}$$

In other words, we define $B_0(g)$ as the smallest constant B in (2.2). Then we get for free that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \le K_n \|\nabla u\|_2^2 + B_0(g)\|u\|_1^2 \tag{2.3}$$

We refer to (2.3) as the saturated form of the sharp inequality (2.2). Taking $u \equiv 1$ in (2.3), it is easily seen that $B_0(g) \ge V_g^{-(n+2)/n}$. In particular, when it exists, $B_0(g)$ depends on the manifold. When (2.3) is true we can define the notion of an extremal function. We say that a nonzero function $u_0 \in H_1^2(M)$ is an extremal function for (2.3) if

$$\|u_0\|_{2^\star}^2 = K_n \|\nabla u_0\|_2^2 + B_0(g)\|u_0\|_1^2$$

In other words, we say that a nonzero function $u_0 \in H_1^2(M)$ is an extremal function for (2.3) if it realizes the equality in (2.3). Then the second question we ask is the following:

Question 2: *Assuming that* (2.2) *is true, does there exist extremal functions for* (2.3) *?*

In the second case, we assume that the manifold we consider is such that (2.2) is false. In other words, we assume that for any $B > 0$, there exists $u \in H_1^2(M)$ which contradicts (2.2). Then we cannot define anymore the notion of an extremal function. However, coming back to the asymptotically sharp inequality (2.1), we can saturate B_ε. Let $B_\varepsilon(g)$ be the smallest B_ε in (2.1) given by

$$B_\varepsilon(g) = \inf \left\{ B_\varepsilon \text{ s.t. (2.1) is true} \right\}$$

Then we get for free that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \le (K_n + \varepsilon)\, \|\nabla u\|_2^2 + B_\varepsilon(g)\|u\|_1^2 \tag{2.4}$$

Since (2.2) is false, we know that $B_\varepsilon(g) \to +\infty$ as $\varepsilon \to 0$. The third and last question we ask is the following:

Question 3: *Assuming that* (2.2) *is false, what is the asymptotic behavior of* $B_\varepsilon(g)$ *as* ε *goes to* 0 *?*

The answers to these three questions involve the dimension and the geometry. Concerning the effect of geometry, we need few words on the Cartan-Hadamard conjecture. This is the subject of the following section.

3 The Cartan-Hadamard conjecture

By definition a Cartan-Hadamard manifold is a complete simply-connected Riemannian manifold of nonpositive sectional curvature. The name of a Cartan-Hadamard manifold comes form the so-called Cartan-Hadamard theorem asserting that for any x in a complete Riemannian manifold of nonpositive sectional curvature, the exponential map $\exp_x$ is a covering. In particular, the exponential map $\exp_x$ realizes a diffeomorphism from $\mathbb{R}^n$ onto $\tilde{M}$ if $\tilde{M}$ is simply-connected.

Let $(\tilde{M}, \tilde{g})$ be a Cartan-Hadamard manifold of dimension n. The n-dimensional Cartan-Hadamard conjecture states that for any smooth bounded domain Ω in $\tilde{M}$,

$$\frac{|\partial\Omega|_{\tilde{g}}}{|\Omega|_{\tilde{g}}^{(n-1)/n}} \ge n \left(\frac{\omega_{n-1}}{n} \right)^{\frac{1}{n}} \tag{3.1}$$

where $|\partial\Omega|_{\tilde{g}}$ is the volume of $\partial\Omega$ with respect to the metric induced by $\tilde{g}$, $|\Omega|_{\tilde{g}}$ is the volume of Ω with respect to $\tilde{g}$, and ω_{n-1} is the volume of the unit $(n-1)$-sphere. Such an inequality holds on the Euclidean space. Moreover, still for the Euclidean space, equality holds if and only if Ω is a ball. Another formulation of the Cartan-Hadamard conjecture is that the sharp Euclidean isoperimetric inequality continues to be true for Cartan-Hadamard manifolds.

The Cartan-Hadamard conjecture is proved to be true in dimension 2 by Weil [39], in dimension 3 by Kleiner [29], and in dimension 4 by Croke [10]. Croke's argument, based on Santalo's formula, is perharps the most surprising. Croke gets explicit Euclidean-type generic Sobolev inequalities for all $n \geq 3$, with the property that one recovers (3.1) only when $n = 4$. For $n \geq 3$, let

$$C(n) = \frac{\omega_{n-2}^{n-2}}{\omega_{n-1}^{n-1}} \left(\int_0^{\pi/2} \cos^{n/(n-2)}(t) \sin^{n-2}(t) dt \right)^{n-2}$$

Croke's result [10] is that for any smooth bounded domain Ω in $\tilde{M}$,

$$\frac{|\partial\Omega|_{\tilde{g}}}{|\Omega|_{\tilde{g}}^{(n-1)/n}} \geq \frac{1}{C(n)^{\frac{1}{n}}}$$

Noting that $C(n)$ is sharp when $n = 4$, it follows that (3.1) is true for any 4-dimensional Cartan-Hadamard manifold.

As far as we know, the Cartan-Hadamard conjecture is open when $n \geq 5$. However, the sharp isoperimetric inequality (3.1) is basically understood for small domains and for large domains. By Yau [40] we indeed have that if $(\tilde{M}, \tilde{g})$ is a Cartan-Hadamard manifold of dimension n, with sectional curvature less than $K < 0$, then for any smooth bounded domain Ω in $\tilde{M}$,

$$|\partial\Omega|_{\tilde{g}} \geq (n-1)\sqrt{-K}|\Omega|_{\tilde{g}}$$

Hence, for such manifolds, (3.1) is true provided that the volume of Ω is sufficiently large. On the other hand, thanks to the recent Druet [14] and Johnson and Morgan [28], we also have curvature conditions which ensure that (3.1) is true if the diameter of Ω is sufficiently small. We refer to these references for more details.

4 Sharp Sobolev-Poincaré inequalities - The results

We return to the sharp Sobolev-Poincaré inequality and to the questions we asked. We start with the first question we asked of whether or not the sharp Sobolev-Poincaré inequality (2.2) is true. A first answer to this question is the following, extracted from Druet-Hebey-Vaugon [19] and Hebey [25].

Theorem 4.1 (Extracted from [19] and [25]) *The sharp Sobolev-Poincaré inequality* (2.2) *is true on any smooth compact Riemannian* 3*-manifold. When* $n \geq 4$*,* (2.2) *is still true on any smooth compact Riemannian n-manifold of negative scalar curvature, but* (2.2) *is false when the scalar curvature of the manifold is positive somewhere.*

This first result clearly illustrates the influence of the dimension and the geometry when studying the sharp Sobolev-Poincaré inequality. When $n = 3$, dimension wins and (2.2) is always true without any kind of assumption on the manifold. When $n \geq 4$, geometry wins and (2.2) is sometimes true and sometimes false, depending on the sign of the scalar curvature. Both phenomena are somehow surprising. For instance, if we consider the standard Sobolev inequality, in other words if we replace in (0.4) the square of the L^1-norm of u by the square of the L^2-norm of u, then, as it was shown by Hebey and Vaugon [27], the corresponding sharp inequality is always true.

Still concerning the first question we asked, a natural additional question to ask with respect to Theorem 4.1 is whether or not (2.2) is still true if we push the curvature from negative values to nonpositive values. In other words, an additional natural question to ask is:

Question 1′: *Is* (2.2) *true on manifolds of nonpositive curvature ?*

It turns out rather quickly that the scalar curvature does not control anymore the situation in this critical limit case. We need more geometric information. This is a typical situation where the Cartan-Hadamard conjecture plays a role. The answer to this question we just asked is given by the following result. It is once more extracted from Druet-Hebey-Vaugon [19] and Hebey [25].

Theorem 4.2 (Extracted from [19] and [25]) *The sharp Sobolev-Poincaré inequality* (2.2) *is true on any smooth compact Riemannian n-manifold, $n \geq 4$, of nonpositive sectional curvature if the n-dimensional Cartan-Hadamard conjecture is true. The sharp Sobolev-Poincaré inequality* (2.2) *is also true on any smooth compact conformally flat Riemannian n-manifold, $n \geq 4$, of nonpositive scalar curvature. On the other hand,* (2.2) *is false if $n \geq 6$, the manifold is not conformally flat and the scalar curvature is zero around one nonconformally flat point.*

Since the 4-dimensional Cartan-Hadamard conjecture is true, it follows from the first part of this theorem that the sharp Sobolev-Poincaré inequality (2.2) is true on any smooth compact Riemannian 4-manifold of nonpositive sectional curvature.

By definition, a Riemannian manifold (M, g) is said to be conformally flat if, up to conformal changes of the metric, we do get local isometries with the Euclidean space. When $n \geq 4$, which is the case in Theorem 4.2, this amounts to say that the Weyl curvature tensor is zero. When this is not the case, we refer to nonconformally flat points as points where the Weyl curvature tensor is not zero.

Theorem 4.2 clearly illustrates the idea that we need more geometric informations when pushing the curvature from negative values to nonpositive values. This is clear in the first part of the theorem where we do need the Cartan-Hadamard conjecture. This is also clear in the second and third parts of the theorem. According to the second part we can push the scalar curvature from negative values to nonpositive values when the manifold is conformally flat. According to the third part, at least when $n \geq 6$, there is no hope that (2.2) is true under the only assumption that the scalar curvature is nonpositive.

A simple corollary to the third part of the theorem is the following rigidity type result. Similar phenomena were observed for the standard Sobolev inequalities by Druet [12] in the compact setting, and by Ledoux [30] in the complete setting.

Corollary 4.1 *Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 6$ and nonnegative Ricci curvature. If* (2.2) *is true on (M,g), then g is flat and M is covered by a torus.*

Theorem 4.2 leaves open the question of whether or not (2.2) is true on manifolds of nonpositive scalar curvature and dimensions 4 and 5 (the 3-dimensional case is settled in Theorem 4.1). The following result answers this question.

Theorem 4.3 (Extracted from [16]) *The sharp Sobolev-Poincaré inequality* (2.2) *is true on any smooth compact Riemannian n-manifold, $n = 4, 5$, of nonpositive scalar curvature.*

Thanks to this theorem, and thanks to Theorems 4.1 and 4.2, we thus face the following situation:

(1) When $n = 3$, (2.2) is true without any assumption on the scalar curvature ;

(2) When $n = 4, 5$, (2.2) is true if the scalar curvature is nonpositive ;

(3) When $n \geq 6$, (2.2) is true if the scalar curvature is nonpositive and the manifold is conformally flat, but there are in any dimensions $n \geq 6$ examples of non conformally flat manifolds of nonpositive scalar curvature for which (2.2) is false.

In particular, corollary 4.1 is false in dimension 3 (thanks to Theorem 4.1), and in dimensions 4 and 5 (thanks to Theorem 4.3).

Theorems 4.1, 4.2 and 4.3 answer the first question we asked in section 2. We are now left with the second and third questions. Namely with the question of the existence of extremal functions for (2.3) when (2.2) is true, and with the question of the asymptotic behavior of (2.4) when (2.2) is false. Thanks to Theorem 4.1 we know that (2.2) is true if the scalar curvature S_g is everywhere negative, and that (2.2) is false if S_g is positive somewhere. The following result, extracted from Druet-Hebey [18] and Hebey [25], answers questions 2 and 3, providing, together with Theorems 4.1, 4.2 and 4.3, a rather complete picture in the study of the sharp Sobolev-Poincaré inequality on compact Riemannian manifolds.

Theorem 4.4 (Extracted from [18] and [25]) *The saturated inequality* (2.3) *possesses extremal functions on any smooth compact Riemannian n-manifold, $n \geq 4$, of negative scalar curvature. On the other hand, if (M,g) is a smooth compact Riemannian n-manifold, $n \geq 4$, whose scalar curvature S_g is positive somewhere, then*

$$B_\varepsilon(g) = C(n) \left(\max_M S_g \right)^{\frac{n+2}{2}} \varepsilon^{-\frac{(n-4)(n+2)}{2(n-2)}} + o\left(\varepsilon^{-\frac{(n-4)(n+2)}{2(n-2)}} \right)$$

where $C(n) > 0$ depending only on n is explicitly known, and where $\varepsilon^{-(n-4)/2}$ has to be understood as $|\ln \varepsilon|$ when $n = 4$.

The constant $C(n)$ in this theorem is given by the following expressions. When $n = 4$ we find that $C(4) = \frac{K_4}{2304\omega_3}$, and when $n \geq 5$, we find that

$$C(n) = \frac{2n(n+2)\omega_n^{2+\frac{4}{n}} K_n^{\frac{n^2-12}{2(n-2)}}}{\left(4^{n-3}n(n-2)(n-4)\right)^{\frac{n+2}{n-2}} \omega_{n-1}^{\frac{2n}{n-2}}}$$

In particular, it follows from the second part of the theorem that for any $C > C(n)$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, and any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \leq (K_n + \varepsilon) \|\nabla u\|_2^2 + C \left(\max_M S_g\right)^3 |\ln \varepsilon|^3 \|u\|_1^2$$

when $n = 4$, and

$$\|u\|_{2^\star}^2 \leq (K_n + \varepsilon) \|\nabla u\|_2^2 + \frac{C \left(\max_M S_g\right)^{\frac{n+2}{2}}}{\varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}}} \|u\|_1^2$$

when $n \geq 5$. The rest of these notes is devoted to the proofs of these results. We follow the original references Druet-Hebey [18], Druet-Hebey-Vaugon [19], and Hebey [25]. In some places, slightly simplier arguments exist thanks to the more recent Druet [14] or Johnson and Morgan [28].

5 Value of the sharp constant

We return to propositions (1) and (2) of section 2, and prove these two propositions. Concerning (1) we may proceed by contradiction. Suppose that there exist a Riemannian n-manifold (M, g) and real numbers $A < K_n$ and B, such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq A \int_M |\nabla u|^2 dv_g + B \left(\int_M |u| dv_g\right)^2 \tag{5.1}$$

Let $x \in M$. It is easy to see that for any $\varepsilon > 0$ there exists a chart (Ω, φ) of M at x, and there exists $\delta > 0$ such that $\varphi(\Omega) = B_0(\delta)$, the Euclidean ball of center 0 and radius δ in $\mathbb{R}^n$, and such that the components g_{ij} of g in this chart satisfy

$$(1 - \varepsilon)\delta_{ij} \leq g_{ij} \leq (1 + \varepsilon)\delta_{ij}$$

as bilinear forms. Choosing ε small enough we then get by (5.1) that there exist $\delta_0 > 0$, $A' < K_n$, and $B' \in \mathbb{R}$ such that for any $\delta \in (0, \delta_0)$ and any $u \in C_0^\infty(B_0(\delta))$,

$$\left(\int_{\mathbb{R}^n} |u|^{2^\star} dx\right)^{2/2^\star} \leq A' \int_{\mathbb{R}^n} |\nabla u|^2 dx + B' \left(\int_{\mathbb{R}^n} |u| dx\right)^2$$

By Hölder,

$$\left(\int_{B_0(\delta)} |u| dx\right)^2 \leq |B_0(\delta)|^{(n+2)/n} \left(\int_{B_0(\delta)} |u|^{2^\star} dx\right)^{2/2^\star}$$

where $|B_0(\delta)|$ denotes the Euclidean volume of $B_0(\delta)$. Choosing δ small enough, it follows that there exist $\delta > 0$ and $A'' < K_n$ such that for any $u \in C_0^\infty(B_0(\delta))$,

$$\left(\int_{\mathbb{R}^n} |u|^{2^\star} dx\right)^{2/2^\star} \leq A'' \int_{\mathbb{R}^n} |\nabla u|^2 dx$$

Let $u \in C_0^\infty(\mathbb{R}^n)$. Set $u_\lambda(x) = u(\lambda x)$, $\lambda > 0$. For λ large enough, $u_\lambda \in C_0^\infty(B_0(\delta))$. Hence,

$$\left(\int_{\mathbb{R}^n} |u_\lambda|^{2^\star} dx\right)^{2/2^\star} \leq A'' \int_{\mathbb{R}^n} |\nabla u_\lambda|^2 dx \tag{5.2}$$

But

$$\int_{I\!R^n} |u_\lambda|^{2^\star} dx = \lambda^{-n} \int_{I\!R^n} |u|^{2^\star} dx$$

while

$$\int_{I\!R^n} |\nabla u_\lambda|^2 dx = \lambda^{2-n} \int_{I\!R^n} |\nabla u|^2 dx$$

so that (5.2) implies that

$$\left(\int_{I\!R^n} |u|^{2^\star} dx\right)^{2/2^\star} \leq A'' \int_{I\!R^n} |\nabla u|^2 dx$$

for all $u \in C_0^\infty(I\!R^n)$. Since $A'' < K_n$, such an inequality is in contradiction with what we said for the Euclidean space. By contradiction, we have proved that any constant A in (0.4) has to be such that $A \geq K_n$. This proves (1).

Concerning (2), we proceed as follows. We fix $\varepsilon > 0$ and let $x \in M$. For any $t > 0$ there exists a chart (Ω, φ) at x such that the components g_{ij} of g in this chart satisfy

$$\frac{1}{1+t}\delta_{ij} \leq g_{ij} \leq (1+t)\delta_{ij}$$

as bilinear forms. Thanks to the sharp Euclidean Sobolev inequality (1.2), choosing $t > 0$ sufficiently small, we can assume that for any smooth function u with compact support in Ω,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq \left(K_n + \frac{\varepsilon}{2}\right) \int_M |\nabla u|^2 dv_g \tag{5.3}$$

Since M is compact, it can be covered by a finite number of such charts (Ω_i, φ_i), $i = 1, \ldots, N$. We let $(\alpha_i)_{i=1,\ldots,N}$ be a smooth partition of unity subordinate to the covering $(\Omega_i)_{i=1,\ldots,N}$, and set

$$\eta_i = \frac{\alpha_i^2}{\sum_j \alpha_j^2}$$

Then $\sqrt{\eta_i}$ is smooth with compact support in Ω_i, and $(\eta_i)_{i=1,\ldots,N}$ is a smooth partition of unity subordinate to the covering $(\Omega_i)_{i=1,\ldots,N}$. For $u \in C^\infty(M)$, we write that

$$\|u\|_{2^\star}^2 = \|u^2\|_{2^\star/2} = \|\sum \eta_i u^2\|_{2^\star/2} \leq \sum \|\eta_i u^2\|_{2^\star/2} = \sum \|\sqrt{\eta_i} u\|_{2^\star}^2$$

Coming back to (5.3), it follows that

$$\begin{aligned}
\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} &\leq \left(K_n + \frac{\varepsilon}{2}\right) \sum_{i=1}^N \int_M \left(\sqrt{\eta_i}|\nabla u| + |\nabla\sqrt{\eta_i}||u|\right)^2 dv_g \\
&= \left(K_n + \frac{\varepsilon}{2}\right) \sum_{i=1}^N \int_M \left(\eta_i |\nabla u|^2 + 2|\nabla\sqrt{\eta_i}|\sqrt{\eta_i}|u||\nabla u| + |\nabla\sqrt{\eta_i}|^2 u^2\right) dv_g
\end{aligned}$$

Writing that for any $\lambda > 0$,

$$2|\nabla u||u| \leq \lambda |\nabla u|^2 + \frac{1}{\lambda} u^2$$

it follows that

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \le \left(K_n + \frac{\varepsilon}{2}\right)\left((1+NH\lambda)\int_M |\nabla u|^2 dv_g + NH(H+\frac{1}{\lambda})\int_M u^2 dv_g\right)$$

where H is such that for any i, $|\nabla\sqrt{\eta_i}| \le H$. Choosing $\lambda > 0$ sufficiently small such that

$$\left(K_n + \frac{\varepsilon}{2}\right)(1+NH\lambda) \le \left(K_n + \frac{2\varepsilon}{3}\right)$$

we get that for any $u \in C^\infty(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \le \left(K_n + \frac{2\varepsilon}{3}\right)\int_M |\nabla u|^2 dv_g + B\int_M u^2 dv_g$$

where

$$B = NH\left(K_n + \frac{\varepsilon}{2}\right)\left(H + \frac{1}{\lambda}\right)$$

Since the embedding $H_1^2 \subset L^2$ is compact, and the embedding $L^2 \subset L^1$ is continuous, it holds that for any $\mu > 0$, there exists $B_\mu > 0$ such that for any $u \in H_1^2(M)$,

$$\|u\|_2^2 \le \mu\|\nabla u\|_2^2 + B_\mu\|u\|_1^2$$

Choosing $\mu > 0$ sufficiently small such that

$$\left(K_n + \frac{2\varepsilon}{3}\right) + B\mu \le K_n + \varepsilon$$

it follows that for any $u \in C^\infty(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \le (K_n+\varepsilon)\int_M |\nabla u|^2 dv_g + \tilde{B}\left(\int_M |u| dv_g\right)^2$$

where $\tilde{B} = BB_\mu$. Since $\varepsilon > 0$ is arbitrary, and since $C^\infty(M)$ is dense in $H_1^2(M)$, we get that for any $\varepsilon > 0$, there exists $B_\varepsilon = \tilde{B} > 0$ such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \le (K_n+\varepsilon)\int_M |\nabla u|^2 dv_g + B_\varepsilon\left(\int_M |u| dv_g\right)^2$$

This proves (2).

6 Test function arguments

We prove in this section the last parts of theorem 4.1 and 4.2. We start with the proof that (2.2) is false if $n \ge 4$ and the scalar curvature S_g is positive somewhere on M. We can do this very simply. Given $x \in M$ such that $S_g(x)$ is positive, we let $r > 0$ be such that $r < i_g(x)$, the injectivity radius at x. In geodesic normal coordinates,

$$\frac{1}{\omega_{n-1}}\int_{S(r)} \sqrt{\det(g_{ij})}ds = 1 - \frac{1}{6n}S_g(x)r^2 + O(r^4)$$

where $S(r)$ stands for the sphere of radius r and center x in M. For $\varepsilon > 0$, we define

$$\begin{aligned} &u_\varepsilon = (\varepsilon + r^2)^{1-n/2} - (\varepsilon + \delta^2)^{1-n/2} \ \text{ if } r \le \delta \\ &u_\varepsilon = 0 \ \text{ otherwise} \end{aligned}$$

where $\delta \in (0, i_g(x))$ is given and $r = d_g(x,.)$. Easy computations lead to

$$\begin{aligned} \int_M |\nabla u_\varepsilon|^2 dv_g &= \frac{(n-2)^2\omega_{n-1}}{2} I_n^{n/2}\ \varepsilon^{1-n/2} \\ &\quad \times\Big(1 - \frac{(n+2)}{6n(n-4)} S_g(x)\ \varepsilon + o(\varepsilon)\Big) \ \text{ if } n > 4 \\ &= \frac{(n-2)^2\omega_{n-1}}{2}\ \varepsilon^{1-n/2} \\ &\quad \times\Big(I_n^{n/2} + \frac{1}{6n} S_g(x)\ \varepsilon\ln\varepsilon + o(\varepsilon\ln\varepsilon)\Big) \ \text{ if } n = 4 \end{aligned}$$

and

$$\begin{aligned} \int_M u_\varepsilon^{2^\star} dv_g &\ge \frac{(n-2)\omega_{n-1}}{2n} I_n^{n/2}\ \varepsilon^{-n/2} \\ &\quad \times\Big(1 - \frac{1}{6(n-2)} S_g(x)\ \varepsilon + o(\varepsilon)\Big) \ \text{ if } n > 4 \\ &\ge \frac{(n-2)\omega_{n-1}}{2n} I_n^{n/2}\ \varepsilon^{-n/2} \\ &\quad \times\Big(1 + o(\varepsilon\ln\varepsilon)\Big) \ \text{ if } n = 4 \end{aligned}$$

where $I_p^q = \int_0^{+\infty}(1+t)^{-p}t^q dt$. As one can easily check

$$\frac{\omega_n}{2^{n-1}\omega_{n-1}} = I_n^{n/2-1} = \frac{(n-2)}{n} I_n^{n/2}$$

Hence,

$$\frac{(n-2)^2\omega_{n-1}}{2}\ I_n^{n/2} = \frac{1}{K_n}\ \Big(\frac{(n-2)\omega_{n-1}}{2n}\ I_n^{n/2}\Big)^{(n-2)/n}$$

Independently,

$$\int_M |u_\varepsilon| dv_g = O(1)$$

so that $\varepsilon^{(n-2)/2}\int_M |u_\varepsilon| dv_g = o(\varepsilon)$ if $n > 4$, and $\varepsilon \int_M |u_\varepsilon| dv_g = o(\varepsilon\ln\varepsilon)$ if $n = 4$. Given $B \in I\!R$, this leads to

$$\begin{aligned} \frac{\|\nabla u_\varepsilon\|_2^2 + B\|u_\varepsilon\|_1^2}{\|u_\varepsilon\|_{2^\star}^2} &\le K_n^{-1}\Big(1 - \frac{S_g(x)}{n(n-4)}\varepsilon + o(\varepsilon)\Big) \ \text{ if } n > 4 \\ &\le K_4^{-1}\Big(1 + \frac{1}{8} S_g(x)\varepsilon\ln\varepsilon + o(\varepsilon\ln\varepsilon)\Big) \ \text{ if } n = 4 \end{aligned}$$

As a consequence, for $n \ge 4$ and any $B \in I\!R$,

$$\frac{\|\nabla u_\varepsilon\|_2^2 + B\|u_\varepsilon\|_1^2}{\|u_\varepsilon\|_{2^\star}^2} < \frac{1}{K_n}$$

provided that $\varepsilon > 0$ is small. Clearly, this implies that (2.2) is false if $n \geq 4$ and the scalar curvature is positive somewhere.

Similarly, we can prove very simply that (2.2) is false if $n \geq 6$, the manifold is not conformally flat and the scalar curvature is zero around one nonconformally flat point. We let W_g be the Weyl tensor of g, and Rc_g be the Ricci curvature of g. By assumption, there exists $x \in M$ such that $W_g(x) \not\equiv 0$ and $S_g \equiv 0$ in $B_x(\delta_0)$ for some $\delta_0 > 0$. We let $\tilde{g}$ be a conformal metric to g such that $Rc_{\tilde{g}}(x) \equiv 0$. We let also $\delta > 0$ be such that $B_x(\delta)$ with respect to $\tilde{g}$ is a subset of $B_x(\delta_0)$ with respect to g. Since the Weyl curvature tensor is a conformal invariant, $W_{\tilde{g}}(x) \not\equiv 0$. Given $B > 0$, it follows from the conformal invariance of the conformal Laplacian that

$$\begin{aligned}
&\inf_{u\in H_1^2(M)\setminus\{0\}} \frac{\int_M |\nabla u|^2 dv_g + B\left(\int_M |u| dv_g\right)^2}{\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star}}\\
&\qquad \leq \inf_{u\in\mathcal{H}} \frac{\int_M |\nabla u|^2 dv_{\tilde{g}} + \frac{n-2}{4(n-1)}\int_M S_{\tilde{g}} u^2 dv_{\tilde{g}} + \hat{B}\left(\int_M |u| dv_{\tilde{g}}\right)^2}{\left(\int_M |u|^{2^\star} dv_{\tilde{g}}\right)^{2/2^\star}}
\end{aligned}$$

where $\hat{B} > 0$, and $\mathcal{H}$ consists of the nonzero functions $u \in H_1^2(M)$ which are such that $\mathrm{Supp} u \subset B_x(\delta)$. For $\varepsilon > 0$, we let u_ε be as above. Then,

$$\int_M |u_\varepsilon| dv_{\tilde{g}} = O(1)$$

so that $\varepsilon^{(n-2)/2}\int_M |u_\varepsilon| dv_g = o(\varepsilon^2)$ if $n > 6$, and $\varepsilon^2 \int_M |u_\varepsilon| dv_g = o(\varepsilon^2 \ln\varepsilon)$ if $n = 6$. It easily follows, as in Aubin [2], that for any $\hat{B} > 0$,

$$\begin{aligned}
&\frac{\int_M |\nabla u_\varepsilon|^2 dv_{\tilde{g}} + \frac{n-2}{4(n-1)}\int_M S_{\tilde{g}} u_\varepsilon^2 dv_{\tilde{g}} + \hat{B}\left(\int_M |u_\varepsilon| dv_{\tilde{g}}\right)^2}{\left(\int_M |u_\varepsilon|^{2^\star} dv_{\tilde{g}}\right)^{2/2^\star}}\\
&\qquad \leq \frac{1}{K_n}\left(1 - C_1 |W_{\tilde{g}}(x)|^2 \varepsilon^2 + o\left(\varepsilon^2\right)\right) \quad \text{if } n > 6\\
&\qquad \leq \frac{1}{K_6}\left(1 + C_2 |W_{\tilde{g}}(x)|^2 \varepsilon^2 \ln\varepsilon + o\left(\varepsilon^2 \ln\varepsilon\right)\right) \quad \text{if } n = 6
\end{aligned}$$

where C_1 and C_2 are explicit positive constants which do not depend on ε. Hence, for any $B > 0$,

$$\inf_{u\in H_1^2(M)\setminus\{0\}} \frac{\int_M |\nabla u|^2 dv_g + B\left(\int_M |u| dv_g\right)^2}{\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star}} < \frac{1}{K_n}$$

and this proves that if $n \geq 6$ and g is scalar flat in an open neighbourhood of one nonconformally flat point, then inequality (2.2) is false.

Corollary 4.1 is an easy consequence of these estimates. We let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 6$ and of nonnegative Ricci curvature. We assume that (2.2) is true on (M, g). The first of the two estimates above gives that the scalar curvature S_g has to be nonpositive. Hence, (M, g) is Ricci flat. This holds as soon as $n \geq 4$. Then the second of the two estimates above gives that g has to be conformally flat. Hence, g is flat, and M is covered by a torus thanks to the Bieberbach theorem.

7 Variational background

For any $\alpha > 0$, we let I_α be the functional defined on $H_1^2(M)\backslash\{0\}$ by

$$I_\alpha(u) = \frac{\|\nabla u\|_2^2 + \alpha\|u\|_1^2}{\|u\|_{2^\star}^2}$$

and let

$$\mu_\alpha = \inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u) \tag{7.1}$$

A reformulation of the results of section 5 is that for any α, $\mu_\alpha \leq K_n^{-1}$, and that for any $\varepsilon > 0$ there exists $\alpha_\varepsilon > 0$ (in spirit large) such that for any $\alpha \geq \alpha_\varepsilon$, $\mu_\alpha \geq (1-\varepsilon)K_n^{-1}$. The result we prove in this section is the following.

Proposition 7.1 *Let (M, g) be a smooth compact Riemannian n-manifold of dimension $n \geq 3$. Suppose that*

$$\inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u) < \frac{1}{K_n} \tag{7.2}$$

Then there exists $u_\alpha \in H_1^2(M)$, $u_\alpha \geq 0$, $u_\alpha \not\equiv 0$, and $\Sigma_\alpha \in L^\infty(M)$ with the property that $0 \leq \Sigma_\alpha \leq 1$ and $\Sigma_\alpha u_\alpha = u_\alpha$, such that

$$\Delta_g u_\alpha + \alpha\Big(\int_M u_\alpha dv_g\Big)\Sigma_\alpha = \mu_\alpha u_\alpha^{2^\star-1} \tag{E_α}$$

and $\int_M u_\alpha^{2^\star} dv_g = 1$. In particular, u_α is a minimizer for μ_α.

The proof of this proposition goes through rather simple arguments. For $q < 2^\star$, let $\theta_q > 1$ be given with the property that θ_q goes to 1 as q goes to $2^\star$. We let $\alpha > 0$ be such that (7.2) is true, and for $q < 2^\star$ we let

$$\lambda_q = \inf_{H_1^2(M)\backslash\{0\}} \frac{\|\nabla u\|_2^2 + \alpha\|u\|_{\theta_q}^2}{\|u\|_q^2}$$

The embedding of $H_1^2(M)$ in $L^q(M)$ being compact, and since the above functional is homogeneous, there exists a nonnegative minimizer u_q for λ_q such that $\|u_q\|_q = 1$. Clearly, u_q is a weak solution of

$$\Delta_g u_q + \alpha\Big(\int_M u_q^{\theta_q} dv_g\Big)^{\frac{2}{\theta_q}-1} u_q^{\theta_q-1} = \lambda_q u_q^{q-1} \tag{7.3}$$

where $\Delta_g = -\text{div}\nabla$ stands for the Laplacian with respect to g. As one can easily check, up to a subsequence, we may assume that for some $\lambda_\alpha \leq \mu_\alpha$, the sequence (λ_q) goes to λ_α as q goes to $2^\star$. Noting that (u_q) is bounded in $H_1^2(M)$, there exists $u_\alpha \in H_1^2(M)$ such that, up to a subsequence, (u_q) converges weakly to u_α in $H_1^2(M)$, strongly to u_α in $L^2(M)$, and almost everywhere. Moreover, one can assume that

$$u_q^{q-1} \rightharpoonup u_\alpha^{2^\star-1} \quad \text{in } L^{2^\sharp}(M)$$

where $2^\sharp = 2n/(n+2)$ is the conjugate exponent of $2^\star$. By (7.2), and since for any $\varepsilon > 0$ there exists B_ε such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \leq (K_n + \varepsilon)\|\nabla u\|_2^2 + B_\varepsilon\|u\|_1^2$$

one has that $u_\alpha \not\equiv 0$. This is by now standard. Let $\varepsilon_q = \theta_q - 1$. Clearly, $(u_q^{\varepsilon_q})$ is bounded in $L^p(M)$ for any $p > 1$. Concerning such an assertion, just note that for $q \gg 1$,

$$\Big(\int_M u_q^{p\varepsilon_q} dv_g\Big)^{1/p} \le \Big(\int_M u_q^2 dv_g\Big)^{\varepsilon_q/2} V_g^{\frac{1}{p}-\frac{\varepsilon_q}{2}} \tag{7.4}$$

where V_g stands for the volume of M with respect to g. Since L^p-spaces are reflexive for $p > 1$, there exists $\Sigma_\alpha \in \bigcap_{p>1} L^p(M)$ such that for any $p > 1$, and up to a subsequence,

$$u_q^{\varepsilon_q} \rightharpoonup \Sigma_\alpha \text{ in } L^p(M)$$

Passing to the limit as q goes to $2^\star$ in (7.4), one gets that for any $p > 1$,

$$\|\Sigma_\alpha\|_p \le V_g^{1/p}$$

As an easy consequence, $\Sigma_\alpha \in L^\infty(M)$ and $0 \le \Sigma_\alpha \le 1$. Another easy claim is that $\Sigma_\alpha \varphi = \varphi$ for any $\varphi \in H_1^2(M)$ having the property that $|\varphi| \le C u_\alpha$ on M for some constant $C > 0$. In particular, $\Sigma_\alpha u_\alpha = u_\alpha$. By passing to the limit in (7.3), one gets that u_α is a weak solution of

$$\Delta_g u_\alpha + \alpha\Big(\int_M u_\alpha dv_g\Big)\Sigma_\alpha = \lambda_\alpha u_\alpha^{2^\star-1} \tag{7.5}$$

Clearly, $\|u_\alpha\|_{2^\star} \le 1$. Mutiplying (7.5) by u_α and integrating over M gives

$$\frac{\|\nabla u_\alpha\|_2^2 + \alpha\|u_\alpha\|_1^2}{\|u_\alpha\|_{2^\star}^2} = \lambda_\alpha \|u_\alpha\|_{2^\star}^{2^\star-2}$$

As one can easily check, this implies that $\|u_\alpha\|_{2^\star} = 1$ and that $\lambda_\alpha = \mu_\alpha$. In particular, u_α is a minimizer for μ_α. This proves Proposition 7.1.

Let $u \in H_1^2(M)$, $u \ge 0$, be such that for any nonnegative $\varphi \in H_1^2(M)$,

$$\int_M (\nabla u \nabla \varphi)\, dv_g \le \int_M u^{2^\star-1}\varphi dv_g$$

where $(\nabla u \nabla \varphi)$ is the pointwise scalar product with respect to g of ∇u and $\nabla \varphi$. We know from PDE theory and the De Giorgi-Nash-Moser iterative scheme that $u \in L^\infty(M)$, with the additional property that for any x in M, any $\Lambda > 0$, any $p > 0$, and any $q > 2^\star$, there exists $\delta > 0$ such that if

$$\int_{B_x(2\delta)} u^q dv_g \le \Lambda$$

then

$$\sup_{y\in B_x(\delta)} u(y) \le \tilde{C}\left(\int_{B_x(2\delta)} u^p dv_g\right)^{1/p}$$

where $\tilde{C} > 0$ does not depend on u. It follows that $u_\alpha \in L^\infty(M)$. In particular, $u_\alpha \in H_2^p(M)$ for any $p > 1$, and it follows from (E_α) that u_α is actually in $C^{1,\lambda}$ for any $\lambda \in (0,1)$. As another remark, the sequence (u_α) is bounded in $H_1^2(M)$.

8 Elementary theory of concentration points

We suppose in this section that the u_α's of section 7 exist for a sequence (α) converging to some $\alpha_0 \in (0,+\infty]$. We assume in what follows that

$$\lim_{\alpha\to\alpha_0} \int_M u_\alpha^2 dv_g = 0 \tag{8.1}$$

As a remark, this is automatically the case if $\alpha_0 = +\infty$. Multiplying (E_α) by u_α, and integrating over M, we get indeed that

$$\|\nabla u_\alpha\|_2^2 + \alpha\|u_\alpha\|_1^2 = \lambda_\alpha$$

As a consequence, $\|u_\alpha\|_1 \to 0$ as $\alpha \to +\infty$, and by Hölder's inequality, since u_α is of norm 1 in $L^{2^\star}$, this implies that $\|u_\alpha\|_2 \to 0$ as $\alpha \to +\infty$. Another remark is the following. By Hebey-Vaugon [27], there exists $B \in I\!R$ such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \le K_n \int_M |\nabla u|^2 dv_g + B\int_M u^2 dv_g$$

Taking $u = u_\alpha$ in this inequality, we get that $1 \le \mu_\alpha K_n + B\int_M u_\alpha^2 dv_g$, and it follows from this inequality and (8.1) that

$$\lim_{\alpha\to\alpha_0} \mu_\alpha = \frac{1}{K_n}$$

Similarly,

$$1 - B\int_M u_\alpha^2 dv_g \le K_n \int_M |\nabla u_\alpha|^2 dv_g = K_n\mu_\alpha - K_n\alpha\left(\int_M u_\alpha dv_g\right)^2$$

and it follows that

$$\lim_{\alpha\to\alpha_0} \alpha\left(\int_M u_\alpha dv_g\right)^2 = 0$$

In particular, the L^1-norm of u_α goes to 0 as α goes to α_0.

Following standard terminology, we say that $x \in M$ is a concentration point for the sequence (u_α) if for any $\delta > 0$,

$$\limsup_{\alpha\to\alpha_0} \int_{B_x(\delta)} u_\alpha^{2^\star} dv_g > 0$$

Since M is compact, the existence of at least such a point is easy to get. We prove the uniqueness of the concentration point in this section.

Proposition 8.1 *Let (M,g) be a smooth compact Riemannian n-manifold of dimension $n \ge 3$. We suppose that* (7.2) *holds for a sequence (α) converging to some $\alpha_0 \in (0,+\infty]$, and we let the u_α's be given by Proposition 7.1. We assume that* (8.1) *holds. Then, up to a subsequence, the sequence (u_α) has one and only one concentration point.*

The proof of this proposition goes through rather simple arguments. Given $x \in M$ and $\delta > 0$, δ small, let $\eta \in C_0^\infty(B_x(\delta))$ be such that $0 \le \eta \le 1$ and $\eta = 1$ in $B_x(\delta/2)$. Multiplying (E_α) by $\eta^2 u_\alpha^k$, $k \ge 1$ real, and integrating over M lead to

$$\int_M \eta^2 u_\alpha^k \Delta_g u_\alpha dv_g + \alpha\left(\int_M u_\alpha dv_g\right)\int_M \eta^2 u_\alpha^k dv_g = \mu_\alpha \int_M \eta^2 u_\alpha^{2^\star+k-1} dv_g \tag{8.2}$$

As one can easily check,

$$\int_M \eta^2 u_\alpha^k \Delta_g u_\alpha dv_g = \frac{4k}{(k+1)^2}\int_M |\nabla(\eta u_\alpha^{(k+1)/2})|^2 dv_g$$
$$-\frac{2(k-1)}{(k+1)^2}\int_M \eta(\Delta_g\eta)u_\alpha^{k+1}dv_g - \frac{2}{k+1}\int_M |\nabla\eta|^2 u_\alpha^{k+1}dv_g$$

while, by Hölder's inequality,

$$\int_M \eta^2 u_\alpha^{2^\star+k-1}dv_g \le \Big(\int_M (\eta u_\alpha^{(k+1)/2})^{2^\star}dv_g\Big)^{2/2^\star}\Big(\int_{B_x(\delta)} u_\alpha^{2^\star}dv_g\Big)^{(2^\star-2)/2^\star}$$

According to Hebey and Vaugon [27], there exists $B > 0$ such that for any $u \in H_1^2(M)$,

$$\Big(\int_M |u|^{2^\star}dv_g\Big)^{2/2^\star} \le K_n\int_M |\nabla u|^2 dv_g + B\int_M u^2 dv_g$$

Coming back to (8.2), and since the second term in the left hand side of (8.2) is nonnegative, one gets that

$$\begin{aligned}A_\alpha(k,\delta)\Big(\int_M (\eta u_\alpha^{(k+1)/2})^{2^\star}dv_g\Big)^{2/2^\star} &\le \frac{k-1}{2k}K_n\int_M \eta(\Delta_g\eta)u_\alpha^{k+1}dv_g\\ +\frac{k+1}{2k}K_n\int_M |\nabla\eta|^2u_\alpha^{k+1}dv_g + B\int_M \eta^2 u_\alpha^{k+1}dv_g&\end{aligned} \tag{8.3}$$

where

$$A_\alpha(k,\delta) = 1 - \frac{(k+1)^2}{4k}\mu_\alpha K_n\Big(\int_{B_x(\delta)} u_\alpha^{2^\star}dv_g\Big)^{(2^\star-2)/2^\star}$$

Suppose now that x is a concentration point for (u_α). Given $\delta > 0$, let

$$\limsup_{\alpha\to\alpha_0}\int_{B_x(\delta)} u_\alpha^{2^\star}dv_g = \lambda_\delta$$

Then $\lambda_\delta > 0$ and $\lambda_\delta \le 1$. Assume that for some $\delta > 0$, $\lambda_\delta < 1$. Together with (7.2), we may then choose $k > 1$ sufficiently close to 1 such that

$$1 - \frac{(k+1)^2}{4k}\mu_\alpha K_n\lambda_\delta^{(2^\star-2)/2^\star} > 0$$

The right hand side of (8.3) being bounded for $k > 1$ close to 1, we get with (8.3) the existence of $K > 0$ such that for $\alpha \gg 1$,

$$\int_M (\eta u_\alpha^{(k+1)/2})^{2^\star}dv_g \le K$$

By Hölder's inequality, writing that $2^\star = (2^\star - k - 1) + (k+1)$,

$$\begin{aligned}\int_{B_x(\delta/2)} u_\alpha^{2^\star}dv_g &\le \Big(\int_M u_\alpha^{2^\star-\frac{2^\star(k-1)}{2^\star-2}}dv_g\Big)^{(2^\star-2)/2^\star}\Big(\int_M (\eta u_\alpha^{(k+1)/2})^{2^\star}dv_g\Big)^{2/2^\star}\\ &\le K^{2/2^\star}\Big(\int_M u_\alpha^{2^\star-\frac{2^\star(k-1)}{2^\star-2}}dv_g\Big)^{(2^\star-2)/2^\star}\end{aligned}$$

Noting that for $k > 1$ close to 1,

$$1 < 2^\star - \frac{2^\star(k-1)}{2^\star - 2} < 2^\star$$

one gets that

$$\lim_{\alpha\to\alpha_0} \int_{B_x(\delta/2)} u_\alpha^{2^\star} dv_g = 0 \tag{8.4}$$

This easily follows from Hölder's inequality since the L^1-norm of u_α goes to 0 as α goes to α_0, and since the $L^{2^\star}$-norm of u_α is 1. Noting that (8.4) is in contradiction with the definition of a concentration point, one actually has that for any $\delta > 0$, $\lambda_\delta = 1$. As one can easily check, up to the extraction of a subsequence, this implies that a concentration point must be unique. Proposition 8.1 is proved.

According to the above proposition, (u_α) has, up to a subsequence, one and only one concentration point x_0. One may then assume that for any $\delta > 0$,

$$\lim_{\alpha\to\alpha_0} \int_{B_{x_0}(\delta)} u_\alpha^{2^\star} dv_g = 1$$

Given $x \neq x_0$, one gets with (8.3) that for $\delta > 0$ small, the $L^{(2^\star)^2/2}$-norm of u_α in $B_x(\delta)$ is bounded. As an easy consequence of the De Giorgi-Nash-Moser iterative scheme, noting that $(2^\star)^2/2 > 2^\star$, we then get that

$$u_\alpha \to 0 \text{ in } C^0_{loc}(M\backslash\{x_0\}) \tag{8.7}$$

as α goes to α_0. The u_α's therefore concentrate in the $L^{2^\star}$-norm at x_0, and they converge C^0 to 0 outside x_0.

9 Localisation for the sharp Sobolev-Poincaré inequality

We prove in this section that (2.2) is localisable. This is the subject of Proposition 9.1. As we will see below, the first part of Theorem 4.2, namely that (2.2) is true for manifolds of nonpositive sectional curvature if the Cartan-Hadamard conjecture is true, is an easy consequence of this proposition.

Proposition 9.1 *Let (M,g) be a smooth compact Riemannian n-manifold of dimension $n \geq 3$. Suppose that for any x in M, there exists Ω_x an open neighborhood of x, and $B_x \in I\!R$, such that for any $u \in C_0^\infty(\Omega_x)$,*

$$\|u\|_{2^\star}^2 \leq K_n \|\nabla u\|_2^2 + B_x \|u\|_1^2 \tag{9.1}$$

Then (2.2) is true on (M,g).

The proof of this proposition goes through rather simple arguments from blow-up theory. For any $\alpha > 0$, we let I_α be the functional of section 7. We assume that (2.2) is locally valid. The proposition reduces to the existence of some α_0 such that

$$\inf_{H_1^2(M)\backslash\{0\}} I_{\alpha_0}(u) \geq \frac{1}{K_n}$$

We proceed by contradiction, and assume that for any $\alpha > 0$,

$$\inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u) < \frac{1}{K_n} \tag{9.2}$$

Then, propositions 7.1 and 8.1 apply. We let x_0 be the concentration point of (u_α), and we use the notations of sections 7 and 8. By assumption there exists $B \in I\!R$ and $\delta > 0$ such that for any $u \in H_{0,1}^2(B_\delta)$,

$$\Big(\int_M |u|^{2^\star} dv_g\Big)^{2/2^\star} \le K_n \int_M |\nabla u|^2 dv_g + B\Big(\int_M |u| dv_g\Big)^2 \tag{9.3}$$

where $B_\delta = B_{x_0}(\delta)$ and $H_{0,1}^2(B_\delta)$ stands for the completion of $C_0^\infty(B_\delta)$ with respect to $\|.\|_{H_1^2}$. We let $\eta \in C_0^\infty(B_\delta)$ be such that $0 \le \eta \le 1$ and $\eta = 1$ in $B_{\delta'}$ for some $\delta' \in (0, \delta)$. Setting $\eta' = 1 - \eta$, (9.3) leads in particular to

$$\Big(\int_{B_{\delta'}} u_\alpha^{2^\star} dv_g\Big)^{2/2^\star} \le K_n \int_M |\nabla((1-\eta')u_\alpha)|^2 dv_g + B\Big(\int_M u_\alpha dv_g\Big)^2$$

Clearly, there exists $C > 0$, independent of α, such that

$$\begin{aligned}\int_M |\nabla((1-\eta')u_\alpha)|^2 dv_g \quad &\le \quad \int_M |\nabla u_\alpha|^2 dv_g + C\int_{M\backslash B_{\delta'}} |\nabla u_\alpha|^2 dv_g \\ &\quad + C\int_{M\backslash B_{\delta'}} u_\alpha |\nabla u_\alpha| dv_g + C\int_{M\backslash B_{\delta'}} u_\alpha^2 dv_g\end{aligned}$$

Multiplying (E_α) by u_α, and integrating over M, gives

$$\int_M |\nabla u_\alpha|^2 dv_g + \alpha\Big(\int_M u_\alpha dv_g\Big)^2 = \mu_\alpha$$

Hence,

$$\begin{aligned}\Big(\int_{B_{\delta'}} u_\alpha^{2^\star} dv_g\Big)^{2/2^\star} \quad &\le \quad K_n\mu_\alpha - \alpha K_n\Big(\int_M u_\alpha dv_g\Big)^2 \\ &\quad + C\int_{M\backslash B_{\delta'}} |\nabla u_\alpha|^2 dv_g + C\int_{M\backslash B_{\delta'}} u_\alpha |\nabla u_\alpha| dv_g \\ &\quad + C\int_{M\backslash B_{\delta'}} u_\alpha^2 dv_g + B\Big(\int_M u_\alpha dv_g\Big)^2\end{aligned}$$

for some other constant $C > 0$ independent of α. Clearly,

$$1 - \Big(\int_{B_{\delta'}} u_\alpha^{2^\star} dv_g\Big)^{2/2^\star} \le \int_{M\backslash B_{\delta'}} u_\alpha^{2^\star} dv_g$$

while

$$\int_{M\backslash B_{\delta'}} u_\alpha |\nabla u_\alpha| dv_g \le \Big(\int_{M\backslash B_{\delta'}} u_\alpha^2 dv_g\Big)^{1/2}\Big(\int_{M\backslash B_{\delta'}} |\nabla u_\alpha|^2 dv_g\Big)^{1/2}$$

Since $\mu_\alpha K_n < 1$, one gets that

$$\begin{aligned}\alpha K_n - B \quad &\le \quad \frac{\int_{M\backslash B_{\delta'}} u_\alpha^{2^\star} dv_g}{\big(\int_M u_\alpha dv_g\big)^2} + C\frac{\int_{M\backslash B_{\delta'}} |\nabla u_\alpha|^2 dv_g}{\big(\int_M u_\alpha dv_g\big)^2} + C\frac{\int_{M\backslash B_{\delta'}} u_\alpha^2 dv_g}{\big(\int_M u_\alpha dv_g\big)^2} \\ &\quad + C\Big(\frac{\int_{M\backslash B_{\delta'}} u_\alpha^2 dv_g}{\big(\int_M u_\alpha dv_g\big)^2}\Big)^{1/2}\Big(\frac{\int_{M\backslash B_{\delta'}} |\nabla u_\alpha|^2 dv_g}{\big(\int_M u_\alpha dv_g\big)^2}\Big)^{1/2}\end{aligned} \tag{9.4}$$

Thanks to the De Giorgi-Nash-Moser iterative scheme,

$$\begin{aligned}\int_{M\backslash B_{\delta'}} u_\alpha^2 dv_g &\leq V_g(\sup_{M\backslash B_{\delta'}} u_\alpha)^2 \\ &\leq C\Big(\int_M u_\alpha dv_g\Big)^2\end{aligned}$$

where V_g stands for the volume of M with respect to g, and $C > 0$ is independent of α. As a consequence,

$$\frac{\int_{M\backslash B_{\delta'}} u_\alpha^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \leq C \tag{9.5}$$

Together with (8.7),

$$\frac{\int_{M\backslash B_{\delta'}} u_\alpha^{2^\star} dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \leq C(\sup_{M\backslash B_{\delta'}} u_\alpha)^{2^\star-2}$$

so that

$$\lim_{\alpha\to+\infty} \frac{\int_{M\backslash B_{\delta'}} u_\alpha^{2^\star} dv_g}{\left(\int_M u_\alpha dv_g\right)^2} = 0 \tag{9.6}$$

For $\delta'' \in (0,\delta')$, let $0 \leq \eta'' \leq 1$ be a smooth function on M such that $\eta'' = 0$ on $B_{\delta''}$ and $\eta'' = 1$ on $M\backslash B_{\delta''}$. Mutiplying (E_α) by $(\eta'')^2 u_\alpha$, and integrating over M, gives

$$\int_M (\eta'')^2 |\nabla u_\alpha|^2 dv_g + 2\int_M \eta'' u_\alpha \langle \nabla\eta'', \nabla u_\alpha\rangle dv_g \leq K_n^{-1}\int_M (\eta'')^2 u_\alpha^{2^\star} dv_g$$

In particular,

$$\begin{aligned}\int_M (\eta'')^2|\nabla u_\alpha|^2 dv_g &\leq C\int_M (\eta'')^2 u_\alpha^{2^\star} dv_g \\ &\quad + C\Big(\int_M |\nabla\eta''|^2 u_\alpha^2 dv_g\Big)^{1/2}\Big(\int_M (\eta'')^2|\nabla u_\alpha|^2 dv_g\Big)^{1/2}\end{aligned}$$

for some constant $C > 0$ independent of α. Hence,

$$\begin{aligned}\frac{\int_M (\eta'')^2|\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} &\leq C\frac{\int_M (\eta'')^2 u_\alpha^{2^\star} dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \\ &\quad + C\Big(\frac{\int_M |\nabla\eta''|^2 u_\alpha^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2}\Big)^{1/2}\Big(\frac{\int_M (\eta'')^2|\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2}\Big)^{1/2}\end{aligned}$$

By (9.6),

$$\lim_{\alpha\to+\infty} \frac{\int_M (\eta'')^2 u_\alpha^{2^\star} dv_g}{\left(\int_M u_\alpha dv_g\right)^2} = 0$$

while by (9.5),

$$\frac{\int_M |\nabla\eta''|^2 u_\alpha^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \leq C$$

for some $C > 0$ independent of α. Noting that

$$\frac{\int_{M\backslash B_{\delta'}} |\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \leq \frac{\int_M (\eta'')^2 |\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2}$$

one gets the existence of $C > 0$ independent of α such that

$$\frac{\int_{M\backslash B_{\delta'}} |\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \leq C \tag{9.7}$$

Combining (9.4) with (9.5) to (9.7), leads to a contradiction. This ends the proof of the proposition.

The first part of Theorem 4.2, namely that (2.2) is true for manifolds of nonpositive sectional curvature if the Cartan-Hadamard conjecture is true, is an easy consequence of Proposition 9.1. Given (M, g) a smooth compact Riemannian n-manifold, we suppose that its sectional curvature K_g is nonpositive, and that the n-dimensional Cartan-Hadamard conjecture is true. Let $(\tilde{M}, \tilde{g})$ be the universal Riemannian covering of (M, g). Then for any smooth bounded domain Ω in $\tilde{M}$,

$$\frac{|\partial\Omega|_{\tilde{g}}}{|\Omega|_{\tilde{g}}^{(n-1)/n}} \geq n \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{n}} \tag{9.8}$$

By standard arguments, see for instance Hebey [24], (9.8) implies that for any $u \in C_0^\infty(\tilde{M})$,

$$\left(\int_{\tilde{M}} |u|^{2^\star} dv_{\tilde{g}}\right)^{2/2^\star} \leq K_n^2 \int_{\tilde{M}} |\nabla u|^2 dv_{\tilde{g}} \tag{9.9}$$

Since (M, g) is locally isometric to $(\tilde{M}, \tilde{g})$, (9.9) implies that (2.2) is locally valid on (M, g). By Proposition 9.1, with $B_x = 0$, this implies that (2.2) is valid on (M, g). As a remark, the same argument leads to the same conclusion if K_g is a nonpositive constant since (9.8) is true for the hyperbolic space and the Euclidean space. As another remark, the same argument leads to the same conclusion if we only assume that a local n-dimensional Cartan-Hadamard conjecture is true on $(\tilde{M}, \tilde{g})$.

10 The 3-dimensional case

We prove in this section the first part of Theorem 4.1, namely that (2.2) is always true in dimension 3. The particular case where the manifold we consider is conformally flat is easy to handle. The result is there a straightforward consequence of the following inequality obtained by Brezis and Nirenberg [7] (see also Brézis and Lieb [6]): for Ω a smooth bounded domain in $\mathbb{R}^3$, and for any $u \in C_0^\infty(\Omega)$,

$$\|u\|_{2^\star}^2 \leq K_3 \|\nabla u\|_2^2 - \lambda |\Omega|^{-2/3} \|u\|_2^2 \tag{10.1}$$

where $|\Omega|$ stands for the Euclidean volume of Ω, and $\lambda > 0$ explicitly known does not depend on Ω. If ξ stands for the Euclidean metric, and (M, g) is conformally flat, then for any x in

M, there exists $r_x > 0$, and φ_x a smooth positive function on M, such that in some chart at x whose domain contains $\Omega_x = B_x(r_x)$, $\xi = \varphi_x^{-4} g$ on Ω_x. As one can easily check, for $u \in C_0^\infty(\Omega_x)$,

$$\int_M |\nabla(u\varphi_x)|^2 dx = \int_M |\nabla u|^2 dv_g + \frac{1}{8}\int_M S_g u^2 dv_g$$

where S_g stands for the scalar curvature of g. Coming back to (10.1), for any $u \in C_0^\infty(\Omega_x)$,

$$\begin{aligned}\Big(\int_M u^6 dv_g\Big)^{1/3} &+ \lambda|\Omega_x|^{-2/3}\int_M \frac{u^2}{\varphi_x^4} dv_g \\ &\leq K_3 \int_M |\nabla u|^2 dv_g + \frac{1}{8} K_3 \int_M S_g u^2 dv_g\end{aligned}$$

Choosing $r_x > 0$ small enough such that

$$\lambda|\Omega_x|^{-2/3}(\max \varphi_x^{-4}) \geq \frac{1}{8} K_3 (\max S_g)$$

one then gets that for any $x \in M$, there exists Ω_x an open neighborhood of x such that for any $u \in C_0^\infty(\Omega_x)$,

$$\Big(\int_M u^6 dv_g\Big)^{1/3} \leq K_3 \int_M |\nabla u|^2 dv_g$$

Thanks to Proposition 9.1, this proves that (2.2) is always true in dimension 3 when the manifold we consider is conformally flat.

Now we prove the first part of Theorem 4.1, namely that (2.2) is always true in dimension 3, in the more difficult case where the manifold is not necessarily conformally flat. We follow the original reference Druet-Hebey-Vaugon [19], but mention that a much simplier argument exists thanks to the more recent Druet [14] or Johnson and Morgan [28]. As when proving that the Sobolev-Poincaré inequality is localisable, we proceed by contradiction. We assume therefore that for any $\alpha > 0$,

$$\inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u) < \frac{1}{K_3} \tag{10.2}$$

where I_α is as in section 7. Hence the results of sections 7 and 8 hold. As in section 7, (10.2) leads to the existence of a minimizer $u_\alpha \in H_1^2(M)$, $u_\alpha \geq 0$ and of norm 1 in $L^6(M)$. If μ_α stands for the above infimum, one has that

$$\Delta_g u_\alpha + \alpha\big(\int_M u_\alpha dv_g\big)\Sigma_\alpha = \mu_\alpha u_\alpha^5 \tag{E_α}$$

where $\Sigma_\alpha \in L^\infty(M)$ is such that $0 \leq \Sigma_\alpha \leq 1$ and $\Sigma_\alpha \varphi = \varphi$ for any $\varphi \in H_1^2(M)$ having the property that $|\varphi| \leq C u_\alpha$ on M for some constant $C > 0$. Moreover, u_α is in $C^{1,\lambda}$ for any $\lambda \in (0,1)$, and the sequence (u_α) is bounded in $H_1^2(M)$. We also have that,

$$\lim_{\alpha\to+\infty} \mu_\alpha = \frac{1}{K_3} \tag{10.3}$$

and

$$\lim_{\alpha\to+\infty} \alpha\|u_\alpha\|_1^2 = 0 \tag{10.4}$$

Moreover, we may assume that (u_α) has one and only one concentration point x_0, we may assume that for any $\delta > 0$,

$$\lim_{\alpha\to+\infty} \int_{B_{x_0}(\delta)} u_\alpha^6 dv_g = 1 \tag{10.5}$$

and we may assume that

$$u_\alpha \to 0 \ \text{ in } C^0_{loc}(M\backslash\{x_0\}) \tag{10.6}$$

as α goes to $+\infty$.

We let $x_\alpha \in M$ and $\lambda_\alpha \in I\!R$ be such that

$$u_\alpha(x_\alpha) = \|u_\alpha\|_\infty = \lambda_\alpha^{-1/2}$$

According to what we just said, $x_\alpha \to x_0$ and $\lambda_\alpha \to 0$ as $\alpha \to +\infty$. By (10.4), noting that

$$1 = \|u_\alpha\|_6^6 \leq \|u_\alpha\|_\infty^5 \|u_\alpha\|_1$$

one gets that

$$\lim_{\alpha\to+\infty} \alpha\lambda_\alpha^{5/2}\|u_\alpha\|_1 = 0 \tag{10.7}$$

The proof now proceeds in several steps.

STEP 1. We claim that for any $R > 0$,

$$\lim_{\alpha\to+\infty} \int_{B_{x_\alpha}(R\lambda_\alpha)} u_\alpha^6 dv_g = 1 - \varepsilon_R \tag{10.8}$$

where $\varepsilon_R > 0$ is such that $\varepsilon_R \to 0$ as $R \to +\infty$. We let $\exp_{x_\alpha}$ be the exponential map at x_α. There clearly exists $\delta > 0$, independent of α, such that for any α, $\exp_{x_\alpha}$ is a diffeomorphism from $B_0(\delta) \subset I\!R^3$ onto $B_{x_\alpha}(\delta)$. For $x \in B_0(\lambda_\alpha^{-1}\delta)$, set

$$\begin{aligned}
&\tilde{g}_\alpha(x) = (\exp_{x_\alpha}^\star g)(\lambda_\alpha x)\\
&\tilde{u}_\alpha(x) = \lambda_\alpha^{1/2} u_\alpha(\exp_{x_\alpha}(\lambda_\alpha x))\\
&\tilde{\Sigma}_\alpha = \Sigma_\alpha(\exp_{x_\alpha}(\lambda_\alpha x))
\end{aligned}$$

As one can easily check,

$$\Delta_{\tilde{g}_\alpha}\tilde{u}_\alpha + \alpha\|u_\alpha\|_1\lambda_\alpha^{5/2}\tilde{\Sigma}_\alpha = \mu_\alpha\tilde{u}_\alpha^5 \tag{$\tilde{E}_\alpha$}$$

Moreover,

$$\tilde{u}_\alpha(0) = \|\tilde{u}_\alpha\|_\infty = 1 \tag{10.9}$$

and if ξ stands for the Euclidean metric of $I\!R^3$,

$$\lim_{\alpha\to+\infty} \tilde{g}_\alpha = \xi \ \text{ in } C^2(K) \tag{10.10}$$

for any compact subset K of $I\!R^3$. By (10.7), (10.9), and theorem 8.24 of Gilbarg-Trudinger [22], $(\tilde{u}_\alpha)$ is equicontinuous on any compact subset of $I\!R^3$. By Ascoli's theorem, one gets the existence of some $\tilde{u} \in C^0(I\!R^3)$ such that for any compact subset K of $I\!R^3$,

$$\lim_{\alpha\to+\infty} \tilde{u}_\alpha = \tilde{u} \ \text{ in } L^\infty(K) \tag{10.11}$$

Clearly, $\tilde{u}(0) = 1$ and $\tilde{u} \not\equiv 0$. An easy assertion to check is that $\tilde{u} \in H_{0,1}^2(I\!R^3)$, where $H_{0,1}^2(I\!R^3)$ stands for the completion of $C_0^\infty(I\!R^3)$ with respect to

$$\|u\|_{H_{0,1}^2} = \sqrt{\int_{I\!R^3} |\nabla u|^2 dx}$$

Indeed, let $\eta \in C_0^\infty(I\!R^3)$, $0 \le \eta \le 1$, be such that $\eta = 1$ in $B_0(\delta/4)$ and $\eta = 0$ in $I\!R^3 \backslash B_0(\delta/2)$. We set $\eta_\alpha(x) = \eta(\mu_\alpha x)$, and

$$\varphi_\alpha(x) = \eta_\alpha(x)\tilde{u}_\alpha(x)$$

Then, $\varphi_\alpha \in C_0^1(I\!R^3)$, and $\varphi_\alpha \to \tilde{u}$ in $L^\infty(K)$ for any compact subset K of $I\!R^3$. Clearly, there exists $C > 0$ such that for any α,

$$\begin{aligned} \|\varphi_\alpha\|_{H_{0,1}^2} &\le C \int_{I\!R^3} |\nabla \varphi_\alpha|^2 dv_{\tilde{g}_\alpha} \\ &\le C \int_{I\!R^3} \eta_\alpha^2 |\nabla \tilde{u}_\alpha|^2 dv_{\tilde{g}_\alpha} + C\lambda_\alpha^2 \int_{I\!R^3} |\nabla \eta(\mu_\alpha x)|^2 \tilde{u}_\alpha^2 dv_{\tilde{g}_\alpha} \end{aligned}$$

On the one hand,

$$\int_{B_0(\delta\lambda_\alpha^{-1})} \tilde{u}_\alpha^2 dv_{\tilde{g}_\alpha} = \lambda_\alpha^{-2} \int_{B_{x_\alpha}(\delta)} u_\alpha^2 dv_g$$

On the other hand,

$$\begin{aligned} \int_{I\!R^3} \eta_\alpha^2 |\tilde{u}_\alpha|^2 dv_{\tilde{g}_\alpha} &\le \int_{B_0(\delta\lambda_\alpha^{-1})} |\tilde{u}_\alpha|^2 dv_{\tilde{g}_\alpha} \\ &= \int_{B_{x_\alpha}(\delta)} |\nabla u_\alpha|^2 dv_g \end{aligned}$$

Hence, (φ_α) is bounded in $H_{0,1}^2(I\!R^3)$, and since $H_{0,1}^2(I\!R^3)$ is reflexive, $\tilde{u} \in H_{0,1}^2(I\!R^3)$. This proves the above assertion. By passing to the limit as α goes to $+\infty$ in $(\tilde{E}_\alpha)$, according to (10.3), (10.7), (10.10), and (10.11), one now gets that $\tilde{u}$ is a solution of

$$\Delta_\xi \tilde{u} = \frac{1}{K_3} \tilde{u}^5 \tag{$\tilde{E}$}$$

By Caffarelli-Gidas-Spruck [8], or also Obata [32],

$$\tilde{u}(x) = \Big(\frac{3K_3}{3K_3 + |x|^2}\Big)^{1/2}$$

since $\tilde{u}(0) = 1$. Noting that $\tilde{u}$ is of norm 1 in $L^6(I\!R^3)$, and that for any $R > 0$,

$$\int_{B_{x_\alpha}(R\lambda_\alpha)} u_\alpha^6 dv_g = \int_{B_0(R)} \tilde{u}_\alpha^6 dv_{\tilde{g}_\alpha}$$

one gets that

$$\lim_{\alpha \to +\infty} \int_{B_{x_\alpha}(R\lambda_\alpha)} u_\alpha^6 dv_g = 1 - \int_{I\!R^3 \backslash B_0(R)} \tilde{u}^6 dx$$

Clearly, this proves (10.8) and the claim we made in step 1.

STEP 2. We claim that there exists $C > 0$, independent of α, such that for any α, and any x,

$$d_g(x_\alpha, x)^{1/2} u_\alpha(x) \le C \tag{10.12}$$

where d_g stands for the distance with respect to g. In order to prove such a claim, set

$$v_\alpha(x) = d_g(x_\alpha, x)^{1/2} u_\alpha(x)$$

and assume by contradiction that for some subsequence,

$$\lim_{\alpha \to +\infty} \|v_\alpha\|_\infty = +\infty \tag{10.13}$$

Let y_α be some point in M where v_α is maximum. By (10.6), $y_\alpha \to x_0$ as $\alpha \to +\infty$, while by (10.13),

$$\lim_{\alpha \to +\infty} \frac{d_g(x_\alpha, y_\alpha)}{\lambda_\alpha} = +\infty \tag{10.14}$$

Fix now $\delta > 0$ small, and set

$$\Omega_\alpha = u_\alpha(y_\alpha)^2 exp_{y_\alpha}^{-1}(B_{x_\alpha}(\delta))$$

For $x \in \Omega_\alpha$, define

$$\tilde{v}_\alpha(x) = u_\alpha(y_\alpha)^{-1} u_\alpha(\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-2}x))$$

and

$$h_\alpha(x) = (\exp_{y_\alpha}^\star g)(u_\alpha(y_\alpha)^{-2}x)$$

Clearly,

$$\lim_{\alpha \to +\infty} h_\alpha = \xi \quad \text{in } C^2(B_0(2)) \tag{10.15}$$

Moreover, as one can easily check,

$$\Delta_{h_\alpha} \tilde{v}_\alpha \le \mu_\alpha \tilde{v}_\alpha^5 \tag{10.16}$$

Since $v_\alpha(y_\alpha)$ goes to $+\infty$ as α goes to $+\infty$, and together with (10.13), one gets that for α large, and all $x \in B_0(2)$,

$$d_g(x_\alpha, exp_{y_\alpha}(u_\alpha(y_\alpha)^{-2}x)) \ge \frac{1}{2} d_g(x_\alpha, y_\alpha) \tag{10.17}$$

This implies that

$$\begin{aligned}
\tilde{v}_\alpha(x) &\le \sqrt{2} d_g(x_\alpha, y_\alpha)^{-1/2} u_\alpha(y_\alpha)^{-1} v_\alpha(exp_{y_\alpha}(u_\alpha(y_\alpha)^{-2}x)) \\
&\le \sqrt{2} d_g(x_\alpha, y_\alpha)^{-1/2} u_\alpha(y_\alpha)^{-1} v_\alpha(y_\alpha)
\end{aligned}$$

so that for α large,

$$\sup_{x \in B_0(2)} \tilde{v}_\alpha(x) \le \sqrt{2} \tag{10.18}$$

By (2.14) and (2.17), given $R > 0$, and for α large,

$$B_{y_\alpha}(2u_\alpha(y_\alpha)^{-2}) \bigcap B_{x_\alpha}(R\lambda_\alpha) = \emptyset$$

Noting that

$$\int_{B_0(2)} \tilde{v}_\alpha^6 dv_{h_\alpha} = \int_{B_{y_\alpha}(2u_\alpha(y_\alpha)^{-2})} u_\alpha^6 dv_g$$

and together with (10.8), one gets that

$$\lim_{\alpha\to+\infty} \int_{B_0(2)} \tilde{v}_\alpha^6 dv_{h_\alpha} = 0 \tag{10.19}$$

By (10.15), (10.16), (10.18), and (10.19), and the De Giorgi-Nash-Moser iterative scheme, one gets that

$$\lim_{\alpha\to+\infty} \sup_{x\in B_0(1)} \tilde{v}_\alpha(x) = 0$$

But $\tilde{v}_\alpha(0) = 1$, so that (10.13) must be false. This proves (10.12) and the claim we made in step 2.

STEP 3. We prove the result, showing that (10.2) leads to a contradiction. We let $\delta > 0$ small to be fixed later on, and for any α, we let $\eta_\alpha \in C_0^\infty(B_{x_\alpha}(4\delta))$ be such that $0 \le \eta_\alpha \le 1$, $\eta_\alpha = 1$ in $B_{x_\alpha}(2\delta)$, and $|\nabla\eta_\alpha| \le C/\delta$. Here, and in what follows, C denotes a constant independent of α and δ. By Brezis and Nirenberg [7], inequality (10.1), and passing through geodesic normal coordinates,

$$\Big(\int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dx\Big)^{1/3} \le K_3 \int_{B_{x_\alpha}(4\delta)} |\nabla(\eta_\alpha u_\alpha)|_\xi^2 dx - \frac{\lambda}{\delta^2}\int_{B_{x_\alpha}(4\delta)} (\eta_\alpha u_\alpha)^2 dx \tag{10.20}$$

where $\lambda > 0$ does not depend on α and δ. When confusions are possible, we write $|.|_\xi$ and $|.|_g$ to specify the metric with respect to which norms are taken. Starting from the Cartan expansion of g in such coordinates,

$$|\nabla(\eta_\alpha u_\alpha)|_\xi^2 \le (1 + Cd_g(x_\alpha, x)^2)|\nabla(\eta_\alpha u_\alpha)|_g^2$$

and

$$(1 - Cd_g(x_\alpha, x)^2)dv_g \le dx \le (1 + Cd_g(x_\alpha, x)^2)dv_g$$

Hence,

$$\int_{B_{x_\alpha}(4\delta)} |\nabla(\eta_\alpha u_\alpha)|_\xi^2 dx \le \int_{B_{x_\alpha}(4\delta)} (1 + Cd_g(x_\alpha, x)^2)|\nabla(\eta_\alpha u_\alpha)|_g^2 dv_g \tag{10.21}$$

On the one hand,

$$\begin{aligned}\int_{B_{x_\alpha}(4\delta)} |\nabla(\eta_\alpha u_\alpha)|_g^2 dv_g \quad &\le \quad \int_M |\nabla u_\alpha|_g^2 dv_g + \frac{C}{\delta^2}\int_{M\backslash B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g \\ &\quad + \frac{C}{\delta}\int_{M\backslash B_{x_\alpha}(2\delta)} u_\alpha |\nabla u_\alpha| dv_g\end{aligned}$$

Multiplying (E_α) by u_α, and integrating over M, gives

$$\begin{aligned}&\int_{B_{x_\alpha}(4\delta)} |\nabla(\eta_\alpha u_\alpha)|_g^2 dv_g \le \mu_\alpha - \alpha\big(\int_M u_\alpha dv_g\big)^2 \\ &\quad + \frac{C}{\delta^2}\int_{M\backslash B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g + \frac{C}{\delta}\int_{M\backslash B_{x_\alpha}(2\delta)} u_\alpha |\nabla u_\alpha|_g dv_g\end{aligned} \tag{10.22}$$

On the other hand,

$$\begin{aligned}\int_{B_{x_\alpha}(4\delta)} d_g(x_\alpha,x)^2|\nabla(\eta_\alpha u_\alpha)|_g^2 dv_g &\le C\int_{M\setminus B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g \\ &+2\int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2|\nabla u_\alpha|_g^2 d_g(x_\alpha,x)^2 dv_g\end{aligned} \tag{10.23}$$

Integrating by parts, and according to (E_α),

$$\begin{aligned}\int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2|\nabla u_\alpha|_g^2 d_g(x_\alpha,x)^2 dv_g &\le C\int_{B_{x_\alpha}(4\delta)} d_g(x_\alpha,x)^2\eta_\alpha^2 u_\alpha^6 dv_g \\ &+C\int_{M\setminus B_{x_\alpha}(2\delta)} u_\alpha|\nabla u_\alpha|_g dv_g + C\int_{M\setminus B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g \\ &+C\int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2 u_\alpha^2 dv_g\end{aligned} \tag{10.24}$$

By (10.12),

$$\int_{B_{x_\alpha}(4\delta)} d_g(x_\alpha,x)^2\eta_\alpha^2 u_\alpha^6 dv_g \le C\int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2 u_\alpha^2 dv_g \tag{10.25}$$

Combining (10.23), (10.24), and (10.25), one may write that

$$\begin{aligned}\int_{B_{x_\alpha}(4\delta)} d_g(x_\alpha,x)^2|\nabla(\eta_\alpha u_\alpha)|_g^2 dv_g &\le C\int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2 u_\alpha^2 dv_g \\ &+\frac{C}{\delta}\int_{M\setminus B_{x_\alpha}(2\delta)} u_\alpha|\nabla u_\alpha|_g dv_g + \frac{C}{\delta^2}\int_{M\setminus B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g\end{aligned} \tag{10.26}$$

Independently,

$$\int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dx \ge \int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dv_g - C\int_{B_{x_\alpha}(2\delta)} d_g(x_\alpha,x)^2 u_\alpha^6 dv_g$$

so that, again by (10.12),

$$\int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dx \ge \int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dv_g - C\int_{B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g$$

For α large, noting that $B_{x_0}(\delta) \subset B_{x_\alpha}(2\delta)$, one gets from (10.5) and the fact that $\|u_\alpha\|_2 \to 0$ as $\alpha \to +\infty$, that the right hand side in this inequality is positive. Since it is also less than 1,

$$\Big(\int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dx\Big)^{1/3} \ge \int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dv_g - C\int_{B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g$$

and

$$\Big(\int_{B_{x_\alpha}(2\delta)} u_\alpha^6 dx\Big)^{1/3} \ge 1 - \int_{M\setminus B_{x_\alpha}(2\delta)} u_\alpha^6 dv_g - C\int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2 u_\alpha^2 dv_g \tag{10.27}$$

By (10.2), (10.20), (10.21), (10.22), (10.26), and (10.27), one gets that

$$\begin{aligned}\alpha K_3\Big(\int_M u_\alpha dv_g\Big)^2 &\le \int_{M\setminus B_{x_\alpha}(2\delta)} u_\alpha^6 dv_g + \frac{C}{\delta^2}\int_{M\setminus B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g \\ &+\frac{C}{\delta}\int_{M\setminus B_{x_\alpha}(2\delta)} u_\alpha|\nabla u_\alpha| dv_g + \Big(C - \frac{\lambda}{\delta^2}\Big)\int_{B_{x_\alpha}(4\delta)} \eta_\alpha^2 u_\alpha^2 dv_g\end{aligned} \tag{10.28}$$

We fix now $\delta > 0$ sufficiently small such that

$$C - \frac{\lambda}{\delta^2} < 0$$

Noting that $B_{x_0}(\delta) \subset B_{x_\alpha}(2\delta)$, and writing by Hölder's inequality that

$$\int_{M\backslash B_{x_\alpha}(2\delta)} u_\alpha |\nabla u_\alpha| dv_g \leq \sqrt{\int_{M\backslash B_{x_\alpha}(2\delta)} u_\alpha^2 dv_g} \sqrt{\int_{M\backslash B_{x_\alpha}(2\delta)} |\nabla u_\alpha|^2 dv_g}$$

one gets with (10.28) the existence of some constant $C > 0$, independent of α, such that

$$\begin{aligned} \alpha K_3 \leq & \frac{\int_{M\backslash B_{x_0}(\delta)} u_\alpha^6 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} + C \frac{\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \\ & + C \Big(\frac{\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \Big)^{1/2} \Big(\frac{\int_{M\backslash B_{x_0}(\delta)} |\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \Big)^{1/2} \end{aligned} \tag{10.29}$$

As in the proof of Proposition 9.1, see (9.5) to (9.7), the right hand side in (10.29) is bounded by some positive constant independent of α. Since the left hand side of (10.29) goes to $+\infty$ as α goes to $+\infty$, we get a contradiction. This proves that (2.2) is true on any 3-dimensional manifold.

11 Negative and nonpositive scalar curvature

We start with the proof of the second part of Theorem 4.1 and of the first part of Theorem 4.4, namely that when $n \geq 4$, (2.2) is true and (2.3) possesses extremal functions on any smooth compact Riemannian n-manifold of negative scalar curvature. Then we prove the second part of Theorem 4.2, namely that when $n \geq 4$, (2.2) is true on any smooth compact conformally flat Riemannian n-manifold of nonpositive scalar curvature.

11.1 Negative scalar curvature

Suppose first that (2.2) is not true. Let $\alpha_0 = +\infty$. Then, for any $\alpha \in (0, \alpha_0)$,

$$\inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u) < \frac{1}{K_n}$$

where I_α is as in section 7. Suppose now that (2.2) is true and let $\alpha_0 = B_0(g) K_n^{-1}$. By the definition of $B_0(g)$, for any $\alpha \in (0, \alpha_0)$,

$$\inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u) < \frac{1}{K_n}$$

By section 7 we get in both cases that there exist $u_\alpha \in H_1^2(M)$ and $\Sigma_\alpha \in L^\infty(M)$, $0 \leq \Sigma_\alpha \leq 1$, such that for a sequence (α), $\alpha < \alpha_0$, converging to α_0,

$$\Delta_g u_\alpha + \alpha \left(\int_M u_\alpha dv_g \right) \Sigma_\alpha = \mu_\alpha u_\alpha^{2^\star - 1} \tag{11.1.1}$$

and

$$\int_M u_\alpha^{2^\star} dv_g = 1 \tag{11.1.2}$$

where $\mu_\alpha < K_n^{-1}$ is the infimum of I_α. Moreover, $\Sigma_\alpha \varphi = \varphi$ for any $\varphi \in H_1^2(M)$ such that $|\varphi| \le C u_\alpha$ for some constant $C > 0$. As already mentioned, if $\alpha_0 = +\infty$, we necessarily have that

$$\lim_{\alpha \to \alpha_0} \int_M u_\alpha^2 dv_g = 0 \tag{11.1.3}$$

On the other hand, let us assume that $\alpha_0 = B_0(g) K_n^{-1}$ and that (11.1.3) does not hold. Then, up to a subsequence, $u_\alpha \rightharpoonup u$ in $H_1^2(M)$ and $u_\alpha \to u$ in $L^2(M)$ as $\alpha \to \alpha_0$, where $u \in H_1^2(M)$, $u \not\equiv 0$. Up to another subsequence, we may also assume that $\mu_\alpha \to \mu$ as $\alpha \to \alpha_0$. We claim now that u is an extremal function for (2.3). Indeed, since $0 \le \Sigma_\alpha \le 1$,

$$\lim_{\alpha \to \alpha_0} \int_M \Sigma_\alpha \left(u_\alpha - u\right) dv_g = 0$$

and we also have that $\int_M \Sigma_\alpha u_\alpha dv_g = \int_M u_\alpha dv_g$ and $\int_M u_\alpha dv_g \to \int_M u dv_g$ as $\alpha \to \alpha_0$. It follows that

$$\lim_{\alpha \to \alpha_0} \int_M \Sigma_\alpha u dv_g = \int_M u dv_g$$

Multiplying (11.1.1) by u, integrating over M, and passing to the limit as $\alpha \to \alpha_0$, we then get that

$$\int_M |\nabla u|^2 dv_g + \alpha_0 \left(\int_M u dv_g\right)^2 = \mu \int_M u^{2^\star} dv_g$$

Hence,

$$\frac{1}{K_n} \le \frac{\int_M |\nabla u|^2 dv_g + \alpha_0 \left(\int_M u dv_g\right)^2}{\left(\int_M u^{2^\star} dv_g\right)^{2/2^\star}} \le \lambda \left(\int_M u^{2^\star} dv_g\right)^{1 - \frac{2}{2^\star}} \tag{11.1.4}$$

Since $\lambda \le K_n^{-1}$ and

$$\int_M u^{2^\star} dv_g \le 1 = \liminf_{\alpha \to \alpha_0} \int_M u_\alpha^{2^\star} dv_g$$

it follows from (11.1.4) that $\mu = K_n^{-1}$ and $\|u\|_{2^\star} = 1$. In particular, u is an extremal function for (1.4), and the above claim is proved. Summarizing, the proof of the second part of Theorem 4.1 and of the first part of Theorem 4.4, namely that when $n \ge 4$, (2.2) is true and (2.3) possesses extremal functions on any smooth compact Riemannian n-manifold of negative scalar curvature, reduces to the proof that (11.1.3) is impossible.

We proceed by contradiction. We let (M, g) be a smooth compact Riemannian n-manifold of negative scalar curvature, $n \ge 4$, and we assume that for any $\alpha \in (0, \alpha_0)$,

$$\inf_{H_1^2(M) \setminus \{0\}} I_\alpha(u) < \frac{1}{K_n} \tag{11.1.5}$$

and that

$$\lim_{\alpha \to \alpha_0} \int_M u_\alpha^2 dv_g = 0 \tag{11.1.6}$$

where $\alpha_0 \in (0, +\infty]$ is either $+\infty$ if we want to prove that (2.2) is true, or $B_0(g) K_n^{-1}$ if we want to prove that (2.3) possesses extremal functions. We split the proof into different steps. The two first steps are the n-dimensional versions of the estimates we proved when dealing with the 3-dimensional case.

By (11.1.5) the results of sections 7 and 8 hold. As in section 7, (11.1.5) leads to the existence of a minimizer $u_\alpha \in H_1^2(M)$, $u_\alpha \geq 0$ and of norm 1 in $L^{2^\star}(M)$. If μ_α stands for the infimum in (11.1.5), one has that

$$\Delta_g u_\alpha + \alpha(\int_M u_\alpha dv_g)\Sigma_\alpha = \mu_\alpha u_\alpha^{2^\star-1} \tag{E_α}$$

where $\Sigma_\alpha \in L^\infty(M)$ is such that $0 \leq \Sigma_\alpha \leq 1$ and $\Sigma_\alpha\varphi = \varphi$ for any $\varphi \in H_1^2(M)$ having the property that $|\varphi| \leq Cu_\alpha$ on M for some constant $C > 0$. Moreover, u_α is in $C^{1,\lambda}$ for any $\lambda \in (0,1)$, and the sequence (u_α) is bounded in $H_1^2(M)$. We also have that,

$$\lim_{\alpha\to\alpha_0} \mu_\alpha = \frac{1}{K_n} \tag{11.1.7}$$

and

$$\lim_{\alpha\to\alpha_0} \alpha\|u_\alpha\|_1^2 = 0 \tag{11.1.8}$$

Moreover, we may assume that (u_α) has one and only one concentration point x_0, we may assume that for any $\delta > 0$,

$$\lim_{\alpha\to\alpha_0} \int_{B_{x_0}(\delta)} u_\alpha^{2^\star} dv_g = 1 \tag{11.1.9}$$

and we may assume that

$$u_\alpha \to 0 \text{ in } C^0_{loc}(M\backslash\{x_0\}) \tag{11.1.10}$$

as α goes to α_0.

We let $x_\alpha \in M$ and $\lambda_\alpha \in I\!R$ be such that

$$u_\alpha(x_\alpha) = \|u_\alpha\|_\infty = \lambda_\alpha^{-(n-2)/2}$$

According to what we just said, $x_\alpha \to x_0$ and $\lambda_\alpha \to 0$ as $\alpha \to \alpha_0$. By (11.1.8), noting that

$$1 = \|u_\alpha\|_{2^\star}^{2^\star} \leq \|u_\alpha\|_\infty^{2^\star-1}\|u_\alpha\|_1$$

one gets that

$$\lim_{\alpha\to\alpha_0} \alpha\lambda_\alpha^{(n+2)/2}\|u_\alpha\|_1 = 0 \tag{11.1.11}$$

As already mentioned, the proof now proceeds in several steps.

STEP 1. We claim that for any $R > 0$,

$$\lim_{\alpha\to\alpha_0} \int_{B_{x_\alpha}(R\lambda_\alpha)} u_\alpha^{2^\star} dv_g = 1 - \varepsilon_R \tag{11.1.12}$$

where $\varepsilon_R > 0$ is such that $\varepsilon_R \to 0$ as $R \to +\infty$. We let $\exp_{x_\alpha}$ be the exponential map at x_α. There clearly exists $\delta > 0$, independent of α, such that for any α, $\exp_{x_\alpha}$ is a diffeomorphism from $B_0(\delta) \subset I\!R^n$ onto $B_{x_\alpha}(\delta)$. For $x \in B_0(\lambda_\alpha^{-1}\delta)$, we set

$$\tilde{g}_\alpha(x) = \left(\exp_{x_\alpha}^\star g\right)(\lambda_\alpha x) \quad , \quad \tilde{u}_\alpha(x) = \lambda_\alpha^{\frac{n-2}{2}} u_\alpha\left(\exp_{x_\alpha}(\lambda_\alpha x)\right)$$

and $\tilde{\Sigma}_\alpha(x) = \Sigma_\alpha\left(\exp_{x_\alpha}(\lambda_\alpha x)\right)$. It is easily seen that

$$\Delta_{\tilde{g}_\alpha}\tilde{u}_\alpha + \alpha\lambda_\alpha^{\frac{n+2}{2}}\left(\int_M u_\alpha dv_g\right)\tilde{\Sigma}_\alpha = \mu_\alpha\tilde{u}_\alpha^{2^\star-1} \tag{11.1.13}$$

Moreover,

$$\tilde{u}_\alpha(0) = \|\tilde{u}_\alpha\|_\infty = 1 \tag{11.1.14}$$

and if ξ stands for the Euclidean metric of $\mathbb{R}^n$,

$$\lim_{\alpha\to\alpha_0}\tilde{g}_\alpha = \xi \quad\text{in } C^2(K) \tag{11.1.15}$$

for any compact subset K of $\mathbb{R}^n$. Thanks to (11.1.11) and (11.1.13)-(11.1.15), we get by standard elliptic theory, as developed in Gilbarg-Trudinger [22], that there exists some $\tilde{u} \in C^1(\mathbb{R}^n)$ such that for any compact subset K of $\mathbb{R}^n$,

$$\lim_{\alpha\to\alpha_0}\tilde{u}_\alpha = \tilde{u} \quad\text{in } C^1(K) \tag{11.1.16}$$

Clearly, $\tilde{u}(0) = 1$ and $\tilde{u} \not\equiv 0$. Moreover, it is easily seen that $\tilde{u} \in H^2_{0,1}(\mathbb{R}^n)$, where $H^2_{0,1}(\mathbb{R}^n)$ is the homogeneous Euclidean Sobolev space of order two for integration and order one for differentiation. By passing to the limit as α goes to α_0 in (11.1.13), according to (11.1.7), (11.1.11), (11.1.15), and (11.1.16), we get that $\tilde{u}$ is a solution of

$$\Delta_\xi\tilde{u} = \frac{1}{K_n}\tilde{u}^{2^\star-1}$$

By Caffarelli-Gidas-Spruck [8], and also Obata [32],

$$\tilde{u}(x) = \left(\frac{1}{1+A|x|^2}\right)^{\frac{n-2}{2}} \tag{11.1.17}$$

where $A^{-1} = n(n-2)K_n$, since $\tilde{u}(0) = \|\tilde{u}\|_\infty = 1$ by (11.1.14) and (11.1.16). Noting that $\tilde{u}$ is of norm 1 in $L^{2^\star}(\mathbb{R}^n)$, and that for any $R > 0$,

$$\int_{B_{x_\alpha}(R\lambda_\alpha)} u_\alpha^{2^\star} dv_g = \int_{B_0(R)} \tilde{u}_\alpha^{2^\star} dv_{\tilde{g}_\alpha}$$

we get that

$$\lim_{\alpha\to\alpha_0}\int_{B_{x_\alpha}(R\lambda_\alpha)} u_\alpha^{2^\star} dv_g = 1 - \int_{\mathbb{R}^n\setminus B_0(R)} \tilde{u}^{2^\star} dx$$

This proves (11.1.12).

STEP 2. We claim that there exists $C > 0$, such that for any α, and any x,

$$d_g(x_\alpha, x)^{\frac{n}{2}-1}u_\alpha(x) \le C \tag{11.1.18}$$

where d_g is the distance with respect to g. In order to prove this claim we set

$$v_\alpha(x) = d_g(x_\alpha, x)^{\frac{n}{2}-1}u_\alpha(x)$$

and assume by contradiction that for some subsequence,

$$\lim_{\alpha\to\alpha_0} \|v_\alpha\|_\infty = +\infty \tag{11.1.19}$$

Let y_α be some point in M where v_α is maximum. By (11.1.10), $y_\alpha \to x_0$ as $\alpha \to \alpha_0$, while by (11.1.19),

$$\lim_{\alpha\to\alpha_0} \frac{d_g(x_\alpha, y_\alpha)}{\lambda_\alpha} = +\infty \tag{11.1.20}$$

Fix now $\delta > 0$ small, and set

$$\Omega_\alpha = u_\alpha(y_\alpha)^{\frac{2}{n-2}} \exp_{y_\alpha}^{-1}\left(B_{x_\alpha}(\delta)\right)$$

For $x \in \Omega_\alpha$, define

$$\tilde{v}_\alpha(x) = u_\alpha(y_\alpha)^{-1} u_\alpha\left(\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n-2}}x)\right)$$

and

$$h_\alpha(x) = \left(\exp_{y_\alpha}^\star g\right)\left(u_\alpha(y_\alpha)^{-\frac{2}{n-2}}x\right)$$

It easily follows from (11.1.19), since M is compact, that $u_\alpha(y_\alpha) \to +\infty$ as $\alpha \to \alpha_0$. Hence,

$$\lim_{\alpha\to\alpha_0} h_\alpha = \xi \quad \text{in } C^2\left(B_0(2)\right) \tag{11.1.21}$$

where ξ is the Euclidean metric. Independently, we have that

$$\Delta_{h_\alpha} \tilde{v}_\alpha \le \mu_\alpha \tilde{v}_\alpha^{2^\star - 1} \tag{11.1.22}$$

Since $v_\alpha(y_\alpha)$ goes to $+\infty$, for α close to α_0, and all $x \in B_0(2)$,

$$d_g\left(x_\alpha, \exp_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n-2}}x)\right) \ge \frac{1}{2} d_g(x_\alpha, y_\alpha) \tag{11.1.23}$$

This implies that

$$\begin{aligned} \tilde{v}_\alpha(x) &\le 2^{\frac{n}{2}-1} d_g(x_\alpha, y_\alpha)^{1-\frac{n}{2}} u_\alpha(y_\alpha)^{-1} v_\alpha\left(\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n-2}}x)\right) \\ &\le 2^{\frac{n}{2}-1} d_g(x_\alpha, y_\alpha)^{1-\frac{n}{2}} u_\alpha(y_\alpha)^{-1} v_\alpha(y_\alpha) \end{aligned}$$

so that for α close to α_0,

$$\sup_{x\in B_0(2)} \tilde{v}_\alpha(x) \le 2^{\frac{n}{2}-1} \tag{11.1.24}$$

By (11.1.20) and (11.1.23), given $R > 0$, and for α close to α_0,

$$B_{y_\alpha}(2u_\alpha(y_\alpha)^{-\frac{2}{n-2}}) \bigcap B_{x_\alpha}(R\lambda_\alpha) = \emptyset \tag{11.1.25}$$

Noting that

$$\int_{B_0(2)} \tilde{v}_\alpha^{2^\star} dv_{h_\alpha} = \int_{B_{y_\alpha}(2u_\alpha(y_\alpha)^{-\frac{2}{n-2}})} u_\alpha^{2^\star} dv_g$$

it follows from (11.1.12) and (11.1.25) that

$$\lim_{\alpha\to\alpha_0} \int_{B_0(2)} \tilde{v}_\alpha^{2^\star} dv_{h_\alpha} = 0 \tag{11.1.26}$$

By (11.1.21), (11.1.22), (11.1.24), (11.1.26), and the De Giorgi-Nash-Moser iterative scheme we get that

$$\lim_{\alpha \to \alpha_0} \sup_{x \in B_0(1)} \tilde{v}_\alpha(x) = 0$$

But $\tilde{v}_\alpha(0) = 1$, so that (11.1.19) must be false. This proves (11.1.18).

STEP 3. We claim that given $R > 0$,

$$\sup_{x \in M \backslash B_{x_\alpha}(R\lambda_\alpha)} d_g(x_\alpha, x)^{\frac{n}{2}-1} u_\alpha(x) = \varepsilon_R(\alpha) \tag{11.1.27}$$

where d_g is the distance with respect to g, and $\lim_{R \to +\infty} \lim_{\alpha \to \alpha_0} \varepsilon_R(\alpha) = 0$. In order to prove this claim we set

$$v_\alpha(x) = d_g(x_\alpha, x)^{\frac{n}{2}-1} u_\alpha(x)$$

and proceed once more by contradiction. Then there exists $y_\alpha \in M$ and $\varepsilon_0 > 0$ such that

$$\lim_{\alpha \to \alpha_0} \frac{d_g(x_\alpha, y_\alpha)}{\lambda_\alpha} = +\infty \text{ and } v_\alpha(y_\alpha) \geq \varepsilon_0$$

As above, we fix $\delta > 0$ small, and set

$$\Omega_\alpha = u_\alpha(y_\alpha)^{\frac{2}{n-2}} \exp_{y_\alpha}^{-1}\left(B_{x_\alpha}(\delta)\right)$$

For $x \in \Omega_\alpha$, we define

$$\tilde{v}_\alpha(x) = u_\alpha(y_\alpha)^{-1} u_\alpha\left(\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n-2}} x)\right)$$

and

$$h_\alpha(x) = \left(\exp_{y_\alpha}^\star g\right)\left(u_\alpha(y_\alpha)^{-\frac{2}{n-2}} x\right)$$

Once again $\Delta_{h_\alpha} \tilde{v}_\alpha \leq \mu_\alpha \tilde{v}_\alpha^{2^\star - 1}$. As when proving (11.1.18), for any $x \in B_0(\frac{1}{2}\varepsilon_0^{\frac{2}{n-2}})$,

$$d_g(x_\alpha, z_\alpha) \geq \frac{1}{2} d_g(x_\alpha, y_\alpha)$$

and

$$\tilde{v}_\alpha(x) = u_\alpha(y_\alpha)^{-1} v_\alpha(z_\alpha) d_g(x_\alpha, z_\alpha)^{1-\frac{n}{2}}$$

where $z_\alpha = \exp_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n-2}} x)$. It follows from (11.1.18) that

$$\tilde{v}_\alpha(x) \leq C 2^{\frac{n}{2}-1} \varepsilon_0^{-1}$$

Noting that for $R > 0$, and for α close to α_0,

$$B_{y_\alpha}\left(\frac{1}{2}\varepsilon_0^{\frac{2}{n-2}} u_\alpha(y_\alpha)^{-\frac{2}{n-2}}\right) \bigcap B_{x_\alpha}(R\lambda_\alpha) = \emptyset$$

we conclude as when proving (11.1.18) that (11.1.27) holds.

STEP 4. We claim that if $\alpha_0 = +\infty$, then, for any $\delta > 0$,

$$\lim_{\alpha\to\alpha_0} \frac{\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_g} = 0 \tag{11.1.28}$$

In other words, we claim that L^2-concentration holds for u_α in any dimension when $\alpha_0 = +\infty$. In order to prove this claim we let $0 \le \eta \le 1$ be a smooth increasing radially symmetric function with respect to x_0 such that $\eta = 1$ in $M\backslash B_{x_0}(\delta)$ and $\eta = 0$ in $B_{x_0}(\delta/2)$. By the De Giorgi-Nash-Moser iterative scheme, and by (11.1.10),

$$\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g \le C_\delta \int_M u_\alpha dv_g \int_M \eta u_\alpha dv_g \tag{11.1.29}$$

where $C_\delta > 0$ is independent of α. Independently, thanks to (E_α), we have that

$$\int_M u_\alpha dv_g \int_M \eta u_\alpha dv_g = \frac{1}{\alpha}\int_M \eta u_\alpha \left(\mu_\alpha u_\alpha^{2^\star-1} - \Delta_g u_\alpha\right) dv_g \tag{11.1.30}$$

Integrating by parts,

$$\int_M \eta u_\alpha \Delta_g u_\alpha dv_g = \int_M \eta |\nabla u_\alpha|^2 dv_g + \int_M (\nabla\eta\nabla u_\alpha)\, u_\alpha dv_g$$

where $(\nabla\eta\nabla u_\alpha)$ is the pointwise scalar product with respect to g of $\nabla\eta$ and ∇u_α. Since $|(\nabla\eta\nabla u_\alpha)| \le |\nabla\eta||\nabla u_\alpha|$, and by Hölder's inequalities,

$$\int_M |(\nabla\eta\nabla u_\alpha)|\, u_\alpha dv_g \le \sqrt{\int_M u_\alpha^2 dv_g}\sqrt{\int_{B_{x_0}(\delta)\backslash B_{x_0}(\delta/2)} |\nabla u_\alpha|^2 dv_g}$$

Coming back to (11.1.29) and (11.1.30), we get that

$$\begin{aligned}\frac{1}{C_\delta}\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g \le{}& \frac{\mu_\alpha}{\alpha}\int_M \eta u_\alpha^{2^\star} dv_g \\ &+ \frac{1}{\alpha}\sqrt{\int_M u_\alpha^2 dv_g}\sqrt{\int_{M\backslash B_{x_0}(\delta/2)} |\nabla u_\alpha|^2 dv_g}\end{aligned}$$

Noting that

$$\int_M \eta u_\alpha^{2^\star} dv_g = \int_M (\eta u_\alpha^{2^\star-2}) u_\alpha^2 dv_g$$

it follows from (11.1.10) that

$$\lim_{\alpha\to+\infty} \frac{\int_M \eta u_\alpha^{2^\star} dv_g}{\int_M u_\alpha^2 dv_g} = 0$$

Then, the proof of (11.1.28) reduces to the proof that for $\hat\delta > 0$ small, there exists $C > 0$, independent of α, such that

$$\int_{M\backslash B_{x_0}(\hat\delta)} |\nabla u_\alpha|^2 dv_g \le C \int_M u_\alpha^2 dv_g \tag{11.1.31}$$

As above, let $0 \leq \eta \leq 1$ be a smooth function such that $\eta = 1$ in $M\backslash B_{x_0}(\hat{\delta})$ and $\eta = 0$ in $B_{x_0}(\hat{\delta}/2)$. Multiplying (E_α) by $\eta^2 u_\alpha$ and integrating over M, we get that

$$\int_M \eta^2 |\nabla u_\alpha|^2 dv_g + 2\int_M \eta u_\alpha \left(\nabla\eta\nabla u_\alpha\right) dv_g \leq \mu_\alpha \int_M \eta^2 u_\alpha^{2^\star} dv_g$$

Therefore,

$$\begin{aligned}\int_M \eta^2 |\nabla u_\alpha|^2 dv_g &\leq C\int_M |\eta\nabla u_\alpha| u_\alpha dv_g + \mu_\alpha \int_M \eta^2 u_\alpha^{2^\star} dv_g \\ &\leq C\sqrt{\int_M u_\alpha^2 dv_g}\sqrt{\int_M \eta^2|\nabla u_\alpha|^2 dv_g} + \mu_\alpha \int_M \eta^2 u_\alpha^{2^\star} dv_g\end{aligned}$$

and we get that

$$\frac{\int_M \eta^2|\nabla u_\alpha|^2 dv_g}{\int_M u_\alpha^2 dv_g} \leq C\sqrt{\frac{\int_M \eta^2|\nabla u_\alpha|^2 dv_g}{\int_M u_\alpha^2 dv_g}} + \mu_\alpha \frac{\int_M \eta^2 u_\alpha^{2^\star} dv_g}{\int_M u_\alpha^2 dv_g}$$

Here again, by (11.1.10),

$$\lim_{\alpha\to+\infty} \frac{\int_M \eta u_\alpha^{2^\star} dv_g}{\int_M u_\alpha^2 dv_g} = 0$$

so that

$$\limsup_{\alpha\to+\infty} \frac{\int_M \eta^2|\nabla u_\alpha|^2 dv_g}{\int_M u_\alpha^2 dv_g} \leq C^2$$

In particular, (11.1.31) holds, and this completes the proof of (11.1.28).

STEP 5. We claim that if $\alpha_0 < +\infty$ and $n \geq 4$, then, for any $\delta > 0$,

$$\lim_{\alpha\to\alpha_0} \frac{\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_g} = 0 \tag{11.1.32}$$

In other words, we claim that L^2-concentration holds for u_α in dimension $n \geq 4$ when $\alpha_0 < +\infty$. In order to prove this claim, we proceed as follows. We clearly have that

$$\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g = \int_{M\backslash B_{x_0}(\delta)} \Sigma_\alpha u_\alpha^2 dv_g$$

Then, by the De Giorgi-Nash-Moser iterative scheme, and by (11.1.10),

$$\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g \leq C\int_M \Sigma_\alpha dv_g \left(\int_M u_\alpha dv_g\right)^2$$

where $C > 0$ is independent of α. Integrating (E_α),

$$\alpha\int_M u_\alpha dv_g \int_M \Sigma_\alpha dv_g = \mu_\alpha \int_M u_\alpha^{2^\star-1} dv_g \tag{11.1.33}$$

By (11.1.33), and (11.1.7), we then get that

$$\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g \leq C\int_M u_\alpha dv_g \int_M u_\alpha^{2^\star-1} dv_g$$

and hence that

$$\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g \le C\sqrt{\int_M u_\alpha^2 dv_g}\int_M u_\alpha^{2^\star-1}dv_g \tag{11.1.34}$$

where $C>0$ is independent of α. First, we suppose that $n\ge 6$. Then $2^\star-1\le 2$, and we get by Hölder's inequalities that

$$\int_M u_\alpha^{2^\star-1}dv_g \le V_g^{\frac{3-2^\star}{2}}\left(\int_M u_\alpha^2 dv_g\right)^{\frac{2^\star-1}{2}}$$

where V_g is the volume of M with respect to g. Coming back to (11.1.34) gives

$$\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g \le C\left(\int_M u_\alpha^2 dv_g\right)^{\frac{2^\star}{2}}$$

Since $2^\star>2$, and $\|u_\alpha\|_2\to 0$, we get that (11.1.32) holds when $n\ge 6$. Now we suppose that $n=5$. Then $2\le 2^\star-1\le 2^\star$, and we get by Hölder's inequalities that

$$\left(\int_M u_\alpha^{2^\star-1}dv_g\right)^{\frac{1}{2^\star-1}} \le \left(\int_M u_\alpha^2 dv_g\right)^{\frac{s}{2}}\left(\int_M u_\alpha^{2^\star}dv_g\right)^{\frac{1-s}{2^\star}}$$

where

$$s=\frac{\frac{1}{2^\star-1}-\frac{1}{2^\star}}{\frac{1}{2}-\frac{1}{2^\star}}=\frac{3}{2(2^\star-1)}$$

Coming back to (11.1.34), and since $\|u_\alpha\|_{2^\star}=1$, we get that

$$\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g \le C\left(\int_M u_\alpha^2 dv_g\right)^{\frac{5}{4}}$$

Here again, $\|u_\alpha\|_2\to 0$. This proves (11.1.32) when $n=5$. At last, we suppose that $n=4$. Then, $2^\star=4$. We have that

$$\begin{aligned}\frac{\int_M u_\alpha^3 dv_g}{\sqrt{\int_M u_\alpha^2 dv_g}} &\le \|u_\alpha\|_{L^\infty(M\backslash B_{x_\alpha}(\delta))}\sqrt{\int_M u_\alpha^2 dv_g}+\frac{\int_{B_{x_\alpha}(\delta)} u_\alpha^3 dv_g}{\sqrt{\int_{B_{x_\alpha}(\delta)} u_\alpha^2 dv_g}}\\ &\le \varepsilon_\alpha+\frac{\int_{B_0(\delta\lambda_\alpha^{-1})}\tilde u_\alpha^3 dv_{\tilde g_\alpha}}{\sqrt{\int_{B_0(\delta\lambda_\alpha^{-1})}\tilde u_\alpha^2 dv_{\tilde g_\alpha}}}\end{aligned}$$

where $\varepsilon_\alpha\to 0$ as $\alpha\to\alpha_0$, and $\tilde u_\alpha$ and $\tilde g_\alpha$ are as in step 1. For any $R>0$, we get by the Cauchy-Schwarz inequality and (11.1.12) that

$$\int_{B_0(\delta\lambda_\alpha^{-1})}\tilde u_\alpha^3 dv_{\tilde g_\alpha}\le\int_{B_0(R)}\tilde u_\alpha^3 dv_{\tilde g_\alpha}+\varepsilon_R\sqrt{\int_{B_0(\delta\lambda_\alpha^{-1})}\tilde u_\alpha^2 dv_{\tilde g_\alpha}}$$

where $\varepsilon_R\to 0$ as $R\to+\infty$. It follows from these equations and (11.1.15) that for any $R>0$,

$$\limsup_{\alpha\to\alpha_0}\frac{\int_M u_\alpha^3 dv_g}{\sqrt{\int_M u_\alpha^2 dv_g}}\le\varepsilon_R+\frac{\int_{\mathbb{R}^n}\tilde u^3 dx}{\sqrt{\int_{B_0(R)}\tilde u^2 dx}}$$

where $\tilde{u}$ is as in (11.1.17). Noting that

$$\lim_{R\to+\infty}\int_{B_0(R)}\tilde{u}^2dx=+\infty$$

when $n=4$, this proves (11.1.32) when $n=4$.

Thanks to the estimates of steps 1 to 5, we are now in position to conclude the proof that when $n\geq 4$, (2.2) is true and (2.3) possesses extremal functions. As already mentioned, the proof reduces to showing that (11.1.5) and (11.1.6) lead to a contradiction.

STEP 6. We claim that (11.1.5) and (11.1.6) lead to a contradiction. For $\delta>0$ small, and $x\in B_0(\delta)$, we set

$$g_\alpha(x)=\exp_{x_\alpha}^\star g(x)\quad\text{and}\quad v_\alpha(x)=u_\alpha\left(\exp_{x_\alpha}(x)\right)$$

We let also η be a smooth cut-off function such that $\eta=1$ in $B_0(\delta/2)$, $\eta=0$ in $I\!R^n\backslash B_0(\delta)$, $|\nabla\eta|\leq C\delta^{-1}$, and $|\nabla^2\eta|\leq C\delta^{-2}$, where, as in what follows, $C>0$ is a constant independent of α and δ. By the definition of K_n,

$$\left(\int_{B_0(\delta)}(\eta v_\alpha)^{2^\star}dx\right)^{\frac{2}{2^\star}}\leq K_n\int_{B_0(\delta)}|\nabla(\eta v_\alpha)|^2dx \tag{11.1.35}$$

We have that

$$\int_{B_0(\delta)}|\nabla(\eta v_\alpha)|^2dx\leq\int_{B_0(\delta)}\eta^2v_\alpha\Delta v_\alpha dx+C\delta^{-2}\int_{B_0(\delta)\backslash B_0(\delta/2)}v_\alpha^2dx$$

and

$$\Delta v_\alpha=\Delta_{g_\alpha}v_\alpha+(g_\alpha^{ij}-\delta^{ij})\partial_{ij}v_\alpha-g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k\partial_kv_\alpha$$

where Δ is the Euclidean Laplacian, δ^{ij} is the Kroenecker symbol, and the $\Gamma(g_\alpha)_{ij}^k$'s are the Christoffel symbols of the Levi-Civita connection with respect to g_α. Hence,

$$\begin{aligned}\int_{B_0(\delta)}|\nabla(\eta v_\alpha)|^2dx&\leq\int_{B_0(\delta)}\eta^2v_\alpha\Delta_{g_\alpha}v_\alpha dx+C\delta^{-2}\int_{B_0(\delta)\backslash B_0(\delta/2)}v_\alpha^2dx\\&+\int_{B_0(\delta)}\eta^2v_\alpha(g_\alpha^{ij}-\delta^{ij})\partial_{ij}v_\alpha dx-\int_{B_0(\delta)}\eta^2v_\alpha g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k\partial_kv_\alpha dx\end{aligned}$$

Integrating by parts, and thanks to (E_α), we then get that

$$\begin{aligned}\int_{B_0(\delta)}|\nabla(\eta v_\alpha)|^2dx&\leq\frac{1}{K_n}\int_{B_0(\delta)}\eta^2v_\alpha^{2^\star}dx-\alpha\int_Mu_\alpha dv_g\int_{B_0(\delta)}\eta^2v_\alpha dx\\&+C\delta^{-2}\int_{B_0(\delta)\backslash B_0(\delta/2)}v_\alpha^2dx-\int_{B_0(\delta)}\eta^2(g_\alpha^{ij}-\delta^{ij})\partial_iv_\alpha\partial_jv_\alpha dx\\&+\frac{1}{2}\int_{B_0(\delta)}\left(\partial_k(g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k)+\partial_{ij}g_\alpha^{ij}\right)\eta^2v_\alpha^2dx\end{aligned}$$

By (11.1.35), this implies in particular that

$$\begin{aligned}0\leq&\int_{B_0(\delta)}\eta^2v_\alpha^{2^\star}dx-\left(\int_{B_0(\delta)}(\eta v_\alpha)^{2^\star}dx\right)^{\frac{2}{2^\star}}\\&+\frac{1}{2}K_n\int_{B_0(\delta)}\left(\partial_k(g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k)+\partial_{ij}g_\alpha^{ij}\right)\eta^2v_\alpha^2dx\\&-K_n\int_{B_0(\delta)}\eta^2(g_\alpha^{ij}-\delta^{ij})\partial_iv_\alpha\partial_jv_\alpha dx+C\delta^{-2}\int_{B_0(\delta)\backslash B_0(\delta/2)}v_\alpha^2dx\end{aligned} \tag{11.1.36}$$

By (11.1.28) and (11.1.32),

$$\lim_{\alpha\to\alpha_0} \frac{\int_{B_0(\delta)\backslash B_0(\delta/2)} v_\alpha^2 dx}{\int_{B_0(\delta)} v_\alpha^2 dx} = 0 \tag{11.1.37}$$

Similarly, since $x_\alpha \to x_0$ as $\alpha \to \alpha_0$, the Cartan expansion of g gives that

$$\lim_{\alpha\to\alpha_0} \left(\partial_k(g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k) + \partial_{ij}g_\alpha^{ij}\right)(0) = \frac{1}{3}S_g(x_0)$$

where S_g is the scalar curvature of g. By (11.1.28) and (11.1.32) we then get that

$$\limsup_{\alpha\to\alpha_0} \frac{\int_{B_0(\delta)} \left(\partial_k(g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k) + \partial_{ij}g_\alpha^{ij}\right)\eta^2 v_\alpha^2 dx}{\int_{B_0(\delta)} v_\alpha^2 dx} = \frac{1}{3}S_g(x_0) + \varepsilon_\delta \tag{11.1.38}$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$. Independently, we claim that when $S_g(x_0) \le 0$,

$$\limsup_{\alpha\to\alpha_0} \frac{\int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star} dx - \left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx\right)^{\frac{2}{2^\star}}}{\int_{B_0(\delta)} v_\alpha^2 dx} \le \varepsilon_\delta \tag{11.1.39}$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$. By Hölder's inequalities we indeed do have that

$$\begin{aligned}&\int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star} dx - \left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx\right)^{\frac{2}{2^\star}}\\ &\le \left(\left(\int_{B_0(\delta)} v_\alpha^{2^\star} dx\right)^{(2^\star-2)/2^\star} - 1\right)\left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx\right)^{2/2^\star}\end{aligned}$$

and thanks to the Cartan expansion of g,

$$dx \le \left(1 + \frac{1}{6}R_{ij}(x_\alpha)x^i x^j + C|x|^3\right) dv_{g_\alpha}$$

where the $R_{ij}(x_\alpha)$'s are the components of the Ricci curvature at x_α in the exponential chart. It follows from these equations that

$$\begin{aligned}&\int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star} dx - \left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx\right)^{\frac{2}{2^\star}}\\ &\le \left(\frac{1+\varepsilon_\delta}{3n}X_\alpha + \varepsilon_\delta \int_{B_0(\delta)} |x|^2 v_\alpha^{2^\star} dv_{g_\alpha}\right)\left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx\right)^{2/2^\star}\end{aligned}$$

where

$$X_\alpha = R_{ij}(x_\alpha)\int_{B_0(\delta)} x^i x^j v_\alpha^{2^\star} dv_{g_\alpha}$$

By (11.1.18) we have that

$$\int_{B_0(\delta)} |x|^2 v_\alpha^{2^\star} dv_{g_\alpha} \le C\int_{B_0(\delta)} v_\alpha^2 dv_{g_\alpha}$$

and by (11.1.27) we have that for any $R > 0$,

$$R_{ij}(x_\alpha) \int_{B_0(\delta)\backslash B_0(R\lambda_\alpha)} x^i x^j v_\alpha^{2^\star} dv_{g_\alpha} \le \varepsilon_R \int_{B_0(\delta)} v_\alpha^2 dv_{g_\alpha}$$

where $\varepsilon_R \to 0$ as $R \to +\infty$. Noting that

$$\frac{R_{ij}(x_\alpha) \int_{B_0(R\lambda_\alpha)} x^i x^j v_\alpha^{2^\star} dv_{g_\alpha}}{\int_{B_0(\delta)} v_\alpha^2 dv_{g_\alpha}} = \frac{R_{ij}(x_\alpha) \int_{B_0(R)} x^i x^j \tilde{u}_\alpha^{2^\star} dv_{\tilde{g}_\alpha}}{\int_{B_0(\delta\lambda_\alpha^{-1})} \tilde{u}_\alpha^2 dv_{g_\alpha}}$$

where $\tilde{u}_\alpha$ and $\tilde{g}_\alpha$ are as in step 1, and that

$$\int_{B_0(R)} x^i x^j \tilde{u}^{2^\star} dx = \frac{1}{n}\delta^{ij} \int_{B_0(R)} |x|^2 \tilde{u}^{2^\star} dx$$

where $\tilde{u}$ is as in (11.1.17) and the δ^{ij}'s are the Kroenecker symbols, we get that

$$\limsup_{\alpha\to\alpha_0} \frac{\frac{1+\varepsilon_\delta}{3n} X_\alpha + \varepsilon_\delta \int_{B_0(\delta)} |x|^2 v_\alpha^{2^\star} dv_{g_\alpha}}{\int_{B_0(\delta)} v_\alpha^2 dv_{g_\alpha}} \le \varepsilon_\delta$$

when $S_g(x_0) \le 0$. By (11.1.10) we also have that $\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx \to 1$ as $\alpha \to \alpha_0$, and, combining these equations, we get (11.1.39). At last, we refer to Druet [15] for details, it can be proved with (11.1.27), (11.1.28) and (11.1.32) that for any $R > 0$,

$$\begin{aligned}\limsup_{\alpha\to\alpha_0} & \frac{\left|\int_{B_0(\delta)} \eta^2 (g_\alpha^{ij} - \delta^{ij}) \partial_i v_\alpha \partial_j v_\alpha dx\right|}{\int_{B_0(\delta)} v_\alpha^2 dx} \\ & \le \varepsilon_R + \varepsilon_\delta + \frac{C}{R^{n-4}} \limsup_{\alpha\to\alpha_0} \frac{1}{\int_{B_0(\delta\lambda_\alpha^{-1})} \tilde{u}_\alpha^2 dv_{\tilde{g}_\alpha}}\end{aligned}$$

where $\varepsilon_R \to 0$ as $R \to +\infty$, $\varepsilon_\delta \to 0$ as $\delta \to 0$, and $\tilde{u}_\alpha$ and $\tilde{g}_\alpha$ are as in step 1. Noting that by (11.1.15) and (11.1.16),

$$\liminf_{\alpha\to\alpha_0} \int_{B_0(\delta\lambda_\alpha^{-1})} \tilde{u}_\alpha^2 dv_{\tilde{g}_\alpha} \ge \int_{B_0(\tilde{R})} \tilde{u}^2 dx$$

for any $\tilde{R} > 0$, where $\tilde{u}$ is as in (11.1.17), and that if $n = 4$,

$$\lim_{\tilde{R}\to+\infty} \int_{B_0(\tilde{R})} \tilde{u}^2 dx = +\infty$$

it follows that

$$\limsup_{\alpha\to\alpha_0} \frac{\int_{B_0(\delta)} \eta^2 (g_\alpha^{ij} - \delta^{ij}) \partial_i v_\alpha \partial_j v_\alpha dx}{\int_{B_0(\delta)} v_\alpha^2 dx} = \varepsilon_\delta \tag{11.1.40}$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$. Combining (11.1.36)-(11.1.40) we then get that

$$\frac{1}{6} K_n S_g(x_0) + \varepsilon_\delta \ge 0 \tag{11.1.41}$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$. Since $S_g(x_0) < 0$, (11.1.41) is impossible, and the contradiction follows. As already mentioned, this proves the second part of Theorem 4.1 and the first part of Theorem 4.4, namely that when $n \ge 4$, (2.2) is true and (2.3) possesses extremal functions on any smooth compact Riemannian n-manifold of negative scalar curvature.

As one can easily check, the above arguments give also that the set of the extremal functions for (2.3) is compact, for instance in the C^1-topology.

11.2 Nonpositive scalar curvature

We prove the second part of Theorem 4.2, namely that when $n \geq 4$, (2.2) is true on any smooth compact conformally flat Riemannian n-manifold of nonpositive scalar curvature. We follow the lines of subsection 11.1 so that we can present L^1-concentration. However, in this specific case, an easier argument exists, based on localisation (Proposition 9.1 above) and conformal invariance (see also Proposition 2.2 in Schoen and Yau [36]). We let (M,g) be a smooth compact conformally flat Riemannian n-manifold of nonpositive scalar curvature, and we assume by contradiction that (11.1.5) holds for all $\alpha \in (0,+\infty)$. Then the results of sections 7 and 8 hold. As in section 7, (11.1.5) leads to the existence of a minimizer $u_\alpha \in H_1^2(M)$, $u_\alpha \geq 0$ and of norm 1 in $L^{2^\star}(M)$. If μ_α stands for the infimum in (11.1.5), one has that

$$\Delta_g u_\alpha + \alpha(\int_M u_\alpha dv_g)\Sigma_\alpha = \mu_\alpha u_\alpha^{2^\star-1} \qquad (E_\alpha)$$

where $\Sigma_\alpha \in L^\infty(M)$ is such that $0 \leq \Sigma_\alpha \leq 1$ and $\Sigma_\alpha\varphi = \varphi$ for any $\varphi \in H_1^2(M)$ having the property that $|\varphi| \leq C u_\alpha$ on M for some constant $C > 0$. Moreover, u_α is in $C^{1,\lambda}$ for any $\lambda \in (0,1)$, and the sequence (u_α) is bounded in $H_1^2(M)$. We also have that,

$$\lim_{\alpha\to+\infty} \mu_\alpha = \frac{1}{K_n} \qquad (11.2.1)$$

and

$$\lim_{\alpha\to+\infty} \alpha\|u_\alpha\|_1^2 = 0 \qquad (11.2.2)$$

Moreover, we may assume that (u_α) has one and only one concentration point x_0, we may assume that for any $\delta > 0$,

$$\lim_{\alpha\to+\infty} \int_{B_{x_0}(\delta)} u_\alpha^{2^\star} dv_g = 1 \qquad (11.2.3)$$

and we may assume that

$$u_\alpha \to 0 \text{ in } C^0_{loc}(M\backslash\{x_0\}) \qquad (11.2.4)$$

as α goes to $+\infty$. As already mentioned, we necessarily have that

$$\lim_{\alpha\to+\infty} \int_M u_\alpha^2 dv_g = 0 \qquad (11.2.5)$$

In particular the estimates of steps 1 to 5 of the preceding subsection hold. In addition to these estimates, we claim that L^1-concentration holds also for the u_α's. In other words, we claim that for any $\delta > 0$,

$$\lim_{\alpha\to+\infty} \frac{\int_{M\backslash B_{x_0}(\delta)} u_\alpha dv_g}{\int_M u_\alpha dv_g} = 0 \qquad (11.2.6)$$

In order to prove this claim we let $0 \leq \eta \leq 1$ be a smooth function such that $\eta = 1$ in $M\backslash B_{x_0}(\delta)$ and $\eta = 0$ in $B_{x_0}(\delta/2)$. We have that

$$\int_M u_\alpha dv_g \int_{M\backslash B_{x_0}(\delta)} u_\alpha dv_g \leq \int_M u_\alpha dv_g \int_M \eta u_\alpha dv_g \qquad (11.2.7)$$

and as when proving the estimate of step 4 of the preceding section, we get by the De Giorgi-Nash-Moser iterative scheme that

$$\begin{aligned}
\int_M u_\alpha dv_g \int_M \eta u_\alpha dv_g &= \frac{1}{\alpha}\int_M \eta u_\alpha \left(\mu_\alpha u_\alpha^{2^\star-1} - \Delta_g u_\alpha\right) dv_g \\
&\leq \frac{1}{\alpha}\mu_\alpha \int_M \eta u_\alpha^{2^\star} dv_g + \frac{1}{\alpha}\int_M \left|(\nabla\eta\nabla u_\alpha)\right| u_\alpha dv_g \\
&\leq \varepsilon_\alpha \left(\int_M u_\alpha dv_g\right)^2 + \frac{C}{\alpha}\sqrt{\int_{M\backslash B_{x_0}(\delta/2)} |\nabla u_\alpha|^2 dv_g}\int_M u_\alpha dv_g
\end{aligned} \tag{11.2.8}$$

where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$, and $C > 0$ is independent of α. The proof of (11.2.6) then reduces to the proof that for $\hat{\delta} > 0$ small, there exists $C > 0$, independent of α, such that

$$\int_{M\backslash B_{x_0}(\hat{\delta})} |\nabla u_\alpha|^2 dv_g \leq C\left(\int_M u_\alpha dv_g\right)^2 \tag{11.2.9}$$

As above, let $0 \leq \eta \leq 1$ be a smooth function such that $\eta = 1$ in $M\backslash B_{x_0}(\hat{\delta})$ and $\eta = 0$ in $B_{x_0}(\hat{\delta}/2)$. Multiplying (E_α) by $\eta^2 u_\alpha$ and integrating over M, we get that

$$\int_M \eta^2|\nabla u_\alpha|^2 dv_g + 2\int_M \eta u_\alpha \left(\nabla\eta\nabla u_\alpha\right) dv_g \leq \mu_\alpha \int_M \eta^2 u_\alpha^{2^\star} dv_g$$

Therefore, by the De Giorgi-Nash-Moser iterative scheme,

$$\begin{aligned}
\int_M \eta^2|\nabla u_\alpha|^2 dv_g &\leq C\int_M |\eta\nabla u_\alpha| u_\alpha dv_g + \mu_\alpha \int_M \eta^2 u_\alpha^{2^\star} dv_g \\
&\leq C\int_M u_\alpha dv_g \sqrt{\int_M \eta^2|\nabla u_\alpha|^2 dv_g} + \mu_\alpha \int_M \eta^2 u_\alpha^{2^\star} dv_g
\end{aligned}$$

and we get that

$$\frac{\int_M \eta^2|\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \leq C\sqrt{\frac{\int_M \eta^2|\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2}} + \mu_\alpha \frac{\int_M \eta^2 u_\alpha^{2^\star} dv_g}{\left(\int_M u_\alpha dv_g\right)^2}$$

By the De Giorgi-Nash-Moser iterative scheme that we apply once again,

$$\lim_{\alpha\to+\infty} \frac{\int_M \eta u_\alpha^{2^\star} dv_g}{\left(\int_M u_\alpha dv_g\right)^2} = 0$$

and it follows that

$$\limsup_{\alpha\to+\infty} \frac{\int_M \eta^2|\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \leq C^2$$

In particular, (11.2.9) holds, and we get from (11.2.7) and (11.2.8) that (11.2.6) holds also. This proves that L^1-concentration holds for the u_α's, and the above claim.

We proceed now with the proof that when $n \geq 4$, (2.2) is true on any smooth compact conformally flat Riemannian n-manifold of nonpositive scalar curvature. In other words, we proceed with the proof that (11.1.5) leads to a contradiction. Since (M, g) is conformally flat, there exists $\varphi \in C^\infty(M)$, $\varphi > 0$, such that $\tilde{g} = \varphi^{4/(n-2)} g$ is flat around x_0. Set $v_\alpha = \varphi^{-1} u_\alpha$. By conformal invariance of the conformal Laplacian,

$$\Delta_{\tilde{g}} v_\alpha + \left(\alpha \int_M u_\alpha dv_g \Sigma_\alpha - \frac{n-2}{4(n-1)} S_g u_\alpha \right) \varphi^{-\frac{n+2}{n-2}} = \mu_\alpha v_\alpha^{2^\star - 1} \tag{11.2.10}$$

We let $\delta > 0$ be such that $\tilde{g}$ is flat in $B_{x_0}(2\delta)$, the ball with respect to $\tilde{g}$ of center x_0 and radius 2δ, and we assimilate $B_{x_0}(2\delta)$ and $\tilde{g}$ with $B_0(2\delta)$ and ξ, where ξ is the Euclidean metric. We let also $0 \le \eta \le 1$ be a smooth cut-off function such that $\eta = 1$ in $B_0(\delta/2)$ and $\eta = 0$ in $\mathbb{R}^n \backslash B_0(\delta)$. By the defintion of K_n,

$$\left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx \right)^{\frac{2}{2^\star}} \le K_n \int_{B_0(\delta)} |\nabla (\eta v_\alpha)|^2 dx \tag{11.2.11}$$

and we have that

$$\int_{B_0(\delta)} |\nabla(\eta v_\alpha)|^2 dx \le \int_{B_0(\delta)} \eta^2 v_\alpha \Delta v_\alpha dx + C \int_{B_0(\delta)\backslash B_0(\delta/2)} v_\alpha^2 dx$$

where $C > 0$ is independent of α. Hence, by (11.2.10), (11.2.11), and since $S_g \le 0$,

$$\begin{aligned} &\left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx \right)^{\frac{2}{2^\star}} + \alpha K_n \int_M u_\alpha dv_g \int_{B_0(\delta)} \eta^2 \varphi^{-2^\star} u_\alpha dx \\ &\le \mu_\alpha K_n \int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star} dx + C \int_{B_0(\delta)\backslash B_0(\delta/2)} v_\alpha^2 dx \end{aligned} \tag{11.2.12}$$

On the one hand, $\mu_\alpha K_n \le 1$. On the other hand, it follows from Hölder's inequalities that

$$\begin{aligned} &\int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star} dx - \left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx \right)^{\frac{2}{2^\star}} \\ &\le \left(\left(\int_{B_0(\delta)} v_\alpha^{2^\star} dx \right)^{1-\frac{2}{2^\star}} - 1 \right) \left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx \right)^{\frac{2}{2^\star}} \end{aligned}$$

Moreover,

$$\int_{B_0(\delta)} v_\alpha^{2^\star} dx = \int_{B_{x_0}(\delta)} u_\alpha^{2^\star} \varphi^{-2^\star} dv_{\tilde{g}} \le \int_M u_\alpha^{2^\star} dv_g = 1$$

and it follows that

$$\mu_\alpha K_n \int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star} dx - \left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx \right)^{\frac{2}{2^\star}} \le 0$$

Coming back to (11.2.12), we then get that

$$\alpha K_n \int_M u_\alpha dv_g \int_{B_0(\delta)} \eta^2 \varphi^{-2^\star} u_\alpha dx \le C \int_{B_0(\delta)\backslash B_0(\delta/2)} v_\alpha^2 dx \tag{11.2.13}$$

We have that

$$\int_{B_0(\delta)} \eta^2 \varphi^{-2^\star} u_\alpha dx \ge \int_{B_{x_0}(\hat{\delta})} u_\alpha dv_g$$

for some $\hat{\delta} > 0$ small, and by the De Giorgi-Nash-Moser iterative scheme, there exists $C > 0$, independent of α, such that for $\hat{\delta} > 0$ sufficiently small,

$$\int_{B_0(\delta)\backslash B_0(\delta/2)} v_\alpha^2 dx \le C \int_M u_\alpha dv_g \int_{M\backslash B_{x_0}(\hat{\delta})} u_\alpha dv_g$$

By (11.2.13) we then get that

$$\alpha K_n \le C \frac{\int_{M\backslash B_{x_0}(\hat{\delta})} u_\alpha dv_g}{\int_{B_{x_0}(\hat{\delta})} u_\alpha dv_g} \tag{11.2.14}$$

while by (11.2.6), the right hand side of (11.2.14) goes to 0 as $\alpha \to +\infty$. A contradiction, so that, as already mentioned, (2.2) is true if (M, g) is conformally flat and the scalar curvature of g is nonpositive.

11.3 Arbitrary energies

In order to fix ideas, we let (T^n, g) be a compact flat torus of dimension $n \ge 3$, and we consider the following equation

$$\Delta_g u + \alpha \|u\|_1 \Sigma = u^{2^\star - 1} \tag{E_α}$$

where $u \in H_1^2(T^n)$, $u \ge 0$, and $\Sigma \in L^\infty(T^n)$, $0 \le \Sigma \le 1$, are such that $\Sigma u = u$, and where $\alpha > 0$. By standard regularity results, if (Σ, u) is a solution of (E_α), then $u \in H_2^p(T^n)$ for all $p > 1$. In particular, $u \in C^{1,\beta}(T^n)$ for all $\beta \in (0, 1)$. We let

$$\mathcal{S}_\alpha = \Big\{(\Sigma_\alpha, u_\alpha) \text{ s.t. } (E_\alpha) \text{ holds}\Big\}$$

and, following Hebey [26], we define the energy function E_m by

$$E_m(\alpha) = \inf_{(\Sigma,u)\in\mathcal{S}_\alpha} \|u\|_{2^\star} \tag{11.3.1}$$

Noting that $\Big(1, (V_g\alpha)^{(n-2)/4}\Big) \in \mathcal{S}_\alpha$, where V_g is the volume of T^n with respect to g, we easily get that $E_m(\alpha) \le V_g^{(n^2-4)/4n} \alpha^{(n-2)/4}$. The above construction can be done on arbitrary compact Riemannian manifolds (M, g). In this case, it easily follows from the test functions arguments developed in section 6 and from Proposition 7.1 that $E_m(\alpha) \le K_n^{-(n-2)/4}$ if $n \ge 4$ and $S_g > 0$ somewhere. We prove here that the following proposition holds, and refer to the remark at the end of this subsection for possible extensions of this result.

Proposition 11.1 *Let (T^n, g) be a compact flat torus of dimension $n \ge 3$. Then*

$$\lim_{\alpha\to+\infty} E_m(\alpha) = +\infty$$

where E_m is the energy function defined in (11.3.1).

Flat torii are interesting since they can be seen as the limit case of manifolds of nonnegative and nonzero curvature, a class for which, as already mentioned, the energy function is bounded. We prove Proposition 11.1 in what follows. We proceed by contradiction, and thus assume that there exists a sequence $(\Sigma_\alpha, u_\alpha)$ in $\mathcal{S}_\alpha$ such that $\|u_\alpha\|_{2^\star} \le \Lambda$ for some $\Lambda > 0$ and all α. Letting $\hat{u}_\alpha = \|u_\alpha\|_{2^\star}^{-1} u_\alpha$, we get that $\|\hat{u}_\alpha\|_{2^\star} = 1$ and that

$$\Delta_g \hat{u}_\alpha + \alpha \|\hat{u}_\alpha\|_1 \Sigma_\alpha = \mu_\alpha \hat{u}_\alpha^{2^\star - 1} \tag{11.3.2}$$

where $\mu_\alpha = \|u_\alpha\|_{2^\star}^{4/(n-2)}$. In particular, $\mu_\alpha \le \Lambda^{4/(n-2)}$. We also have that $\Sigma_\alpha \hat{u}_\alpha = \hat{u}_\alpha$. As in section 8, following standard terminology, we say that $x \in T^n$ is a concentration point for the sequence $(\Sigma_\alpha, \hat{u}_\alpha)$ if for any $\delta > 0$,

$$\limsup_{\alpha \to +\infty} \int_{B_x(\delta)} \hat{u}_\alpha^{2^\star} dv_g > 0$$

Multiplying $(\hat{E}_\alpha)$ by $\hat{u}_\alpha$, and integrating over T^n, we see that $\|\hat{u}_\alpha\|_1 \to 0$ as $\alpha \to +\infty$. It follows that $(\Sigma_\alpha, \hat{u}_\alpha)$ has at least one concentration point. We let $\mathcal{S}$ be the set of the concentration points for $(\Sigma_\alpha, \hat{u}_\alpha)$. Mimicking what we did in section 8, another possible reference is Druet-Hebey-Vaugon [20], it is easily seen that the two following propositions hold: up to a subsequence,

$$\begin{aligned} &\mathcal{S} = \{x_1, \dots, x_p\} \text{ is finite, and} \\ &\hat{u}_\alpha \to 0 \text{ in } C^0_{loc}(T^n \backslash \mathcal{S}). \end{aligned} \tag{11.3.3}$$

In a similar way, mimicking what we did in subsection 11.2, we easily get that for any $\delta > 0$,

$$\lim_{\alpha \to +\infty} \frac{\int_{T^n \backslash \mathcal{B}_\delta} \hat{u}_\alpha dv_g}{\int_{T^n} \hat{u}_\alpha dv_g} = 0 \tag{11.3.4}$$

where $\mathcal{B}_\delta$ is the union of the $B_{x_i}(\delta)$'s, $i = 1, \dots, p$. We also get that

$$\int_{T^n \backslash \mathcal{B}_\delta} \left(|\nabla \hat{u}_\alpha|^2 + \hat{u}_\alpha^2 \right) dv_g \le C \left(\int_{T^n} \hat{u}_\alpha dv_g \right)^2 \tag{11.3.5}$$

where $C > 0$ is independent of α. Concerning terminology, we refer to (11.3.4) as global L^1-concentration. Now we fix $x_i \in \mathcal{S}$. Since g is flat, we can assimilate g with the Euclidean metric ξ around x_i. Given $\delta > 0$ sufficiently small, we let $0 \le \eta \le 1$ be a smooth cut-off function such that $\eta = 1$ in $B_0(\delta/2)$ and $\eta = 0$ in $\mathbb{R}^n \backslash B_0(\delta)$. Thanks to the Pohozaev identity [33],

$$2 \int_{\mathbb{R}^n} \left(x^k \partial_k (\eta \hat{u}_\alpha) \right) \Delta(\eta \hat{u}_\alpha) dx + (n-2) \int_{\mathbb{R}^n} \eta \hat{u}_\alpha \Delta(\eta \hat{u}_\alpha) dx \le 0 \tag{11.3.6}$$

Such an equation makes sense since $\hat{u}_\alpha \in C^{1,\beta}$, $0 < \beta < 1$, and $\hat{u}_\alpha \in H^p_{2,loc}$, $p > 1$. Integrating by parts, we easily get that

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(x^k \partial_k (\eta \hat{u}_\alpha) \right) \Delta(\eta \hat{u}_\alpha) dx = \int_{\mathbb{R}^n} \eta^2 \left(x^k \partial_k \hat{u}_\alpha \right) \Delta \hat{u}_\alpha dx + \mathcal{R}(\alpha) \\ &\int_{\mathbb{R}^n} \eta \hat{u}_\alpha \Delta(\eta \hat{u}_\alpha) dx = \int_{\mathbb{R}^n} \eta^2 \hat{u}_\alpha \Delta \hat{u}_\alpha dx + \mathcal{R}_1(\alpha) \end{aligned} \tag{11.3.7}$$

where

$$|\mathcal{R}_1(\alpha)| \le C_1 \int_{B_0(\delta) \backslash B_0(\delta/2)} |\nabla \hat{u}_\alpha|^2 dx + C_2 \int_{B_0(\delta) \backslash B_0(\delta/2)} \hat{u}_\alpha^2 dx \tag{11.3.8}$$

and $C_1, C_2 > 0$ are independent of α. We have that $\Sigma_\alpha \hat{u}_\alpha = \hat{u}_\alpha$. Thus, $\Sigma_\alpha = 1$ when $\hat{u}_\alpha > 0$. Let f be a smooth function with compact support in $B_0(2\delta)$. Noting that the set of the x's which are such that $\hat{u}_\alpha(x) = 0$ and $|\nabla \hat{u}_\alpha|(x) \neq 0$ is a hypersurface in T^n, and thus of measure zero, we can write that for any k,

$$\begin{aligned}\int_{\{|\nabla \hat{u}_\alpha| \neq 0\}} f(\partial_k \hat{u}_\alpha)\Sigma_\alpha dx &= \int_{\{\hat{u}_\alpha > 0, |\nabla \hat{u}_\alpha| \neq 0\}} f(\partial_k \hat{u}_\alpha)\Sigma_\alpha dx \\ &= \int_{\{\hat{u}_\alpha > 0, |\nabla \hat{u}_\alpha| \neq 0\}} f(\partial_k \hat{u}_\alpha) dx \\ &= \int_{\{|\nabla \hat{u}_\alpha| \neq 0\}} f(\partial_k \hat{u}_\alpha) dx\end{aligned}$$

It follows that for any smooth function f with compact support in $B_0(2\delta)$, and for any k,

$$\int_{\mathbb{R}^n} f(\partial_k \hat{u}_\alpha)\Sigma_\alpha dx = \int_{\mathbb{R}^n} f(\partial_k \hat{u}_\alpha) dx \tag{11.3.9}$$

Thanks to (11.3.2) and (11.3.9),

$$\int_{\mathbb{R}^n} \eta^2 \left(x^k \partial_k \hat{u}_\alpha\right) \Delta \hat{u}_\alpha dx = \mu_\alpha \int_{\mathbb{R}^n} \eta^2 (x^k \partial_k \hat{u}_\alpha) \hat{u}_\alpha^{2^\star - 1} dx - \alpha \int_{T^n} \hat{u}_\alpha dv_g \int_{\mathbb{R}^n} \eta^2 (x^k \partial_k \hat{u}_\alpha) dx$$

while, thanks to (11.3.2),

$$\int_{\mathbb{R}^n} \eta^2 \hat{u}_\alpha \Delta \hat{u}_\alpha dx = \mu_\alpha \int_{\mathbb{R}^n} \eta^2 \hat{u}_\alpha^{2^\star} dx - \alpha \int_{T^n} \hat{u}_\alpha dv_g \int_{\mathbb{R}^n} \eta^2 \hat{u}_\alpha dx$$

Integrating by parts,

$$\int_{\mathbb{R}^n} \eta^2 (x^k \partial_k \hat{u}_\alpha) \hat{u}_\alpha^{2^\star - 1} dx = -\frac{2}{2^\star} \int_{\mathbb{R}^n} \eta (x^k \partial_k \eta) \hat{u}_\alpha^{2^\star} dx - \frac{n-2}{2} \int_{\mathbb{R}^n} \eta^2 \hat{u}_\alpha^{2^\star} dx$$

and

$$\int_{\mathbb{R}^n} \eta^2 (x^k \partial_k \hat{u}_\alpha) dx = -2 \int_{\mathbb{R}^n} \eta (x^k \partial_k \eta) \hat{u}_\alpha dx - n \int_{\mathbb{R}^n} \eta^2 \hat{u}_\alpha dx$$

Combining these equations, and thanks to (11.3.7), we get that

$$\begin{aligned}&2 \int_{\mathbb{R}^n} \left(x^k \partial_k (\eta \hat{u}_\alpha)\right) \Delta (\eta \hat{u}_\alpha) dx + (n-2) \int_{\mathbb{R}^n} \eta \hat{u}_\alpha \Delta (\eta \hat{u}_\alpha) dx \\ &= (n+2)\alpha \int_{T^n} \hat{u}_\alpha dv_g \int_{\mathbb{R}^n} \eta^2 \hat{u}_\alpha dx + \mathcal{R}_1(\alpha) + \alpha \mathcal{R}_2(\alpha) \int_{T^n} \hat{u}_\alpha dv_g\end{aligned}$$

where $\mathcal{R}_1(\alpha)$ is as in (11.3.8), and where, thanks to (11.3.3) and the De Giorgi-Nash-Moser iterative scheme, $\mathcal{R}_2(\alpha)$ is such that

$$|\mathcal{R}_2(\alpha)| \leq C_3 \int_{B_0(\delta) \backslash B_0(\delta/2)} \hat{u}_\alpha dx$$

where $C_3 > 0$ is independent of α. Coming back to (11.3.6), summing over the x_i's in $\mathcal{S}$, and thanks to (11.3.5), we have proved that for $\delta > 0$ sufficiently small,

$$\alpha \int_{T^n} \hat{u}_\alpha dv_g \int_{\mathcal{B}_\delta} \hat{u}_\alpha dv_g \leq C_4 \left(\int_{T^n} \hat{u}_\alpha dv_g\right)^2 + C_5 \alpha \int_{T^n} \hat{u}_\alpha dv_g \int_{T^n \backslash \mathcal{B}_\delta} \hat{u}_\alpha dv_g \tag{11.3.10}$$

where $C_4, C_5 > 0$ are independent of α. In particular, it follows from (11.3.10) that

$$\alpha \frac{\int_{\mathcal{B}_\delta} \hat{u}_\alpha dv_g}{\int_{T^n} \hat{u}_\alpha dv_g} \leq C_4 + C_5 \alpha \frac{\int_{T^n \setminus \mathcal{B}_\delta} \hat{u}_\alpha dv_g}{\int_{T^n} \hat{u}_\alpha dv_g}$$

Letting $\alpha \to +\infty$ we then get our contradiction thanks to (11.3.4). This proves Proposition 11.1.

The above proof extends to compact conformally flat Riemannian manifolds which are scalar flat. A possible example of such a manifold, which is not a flat torus, neither the quotient of a flat torus, consists in the product of the unit sphere with a compact hyperbolic space of the same dimension. In particular, if $\mathcal{M}$ is the class of compact Riemannian manifolds of nonnegative and nonzero curvature, and if $\partial_{C^2}\mathcal{M}$ is its boundary with respect to the C^2-topology, then E_m is (uniformly) bounded in $\mathcal{M}$, but unbounded on $\partial_{C^2}\mathcal{M}$. Independently, using global L^2-concentration instead of global L^1-concentration, and conformal invariance, it is easily checked that a slight modification of the above proof gives that the conclusion of Proposition 11.1 still holds for compact conformally flat Riemannian manifolds of dimension $n \geq 3$ and negative scalar curvature. Open questions on E_m can be found in Hebey [26].

12 Asymptotics when the scalar curvature is positive somewhere

We prove the second part of Theorem 4.4. This is by far the most difficult part in Theorems 4.1 to 4.4. We separate this section into three subsections. The first subsection concerns the study of a closely related problem in the Euclidean context. A test function type argument, based on what is proved in subsection 12.1, is developed in subsection 12.2. The general case of an arbitrary compact Riemannian manifold is treated in subsection 12.3.

12.1 The Euclidean case

Let $\mathcal{B}$ be the unit ball in $I\!R^n$, $n \geq 4$, and $\Delta = -\text{div}(\nabla)$ be the Euclidean Laplacian. We let $C_0^\infty(\mathcal{B})$ be the set of smooth functions with compact support in $\mathcal{B}$, and $H_{0,1}^2(\mathcal{B})$ be the standard Sobolev space defined as the completion of $C_0^\infty(\mathcal{B})$ with respect to the norm $\|u\| = \|\nabla u\|_2$. Given $\alpha > 0$ and $B > 0$, we define λ_B by

$$\lambda_B = \inf_{u \in C_0^\infty(\mathcal{B}) \setminus \{0\}} \frac{\|\nabla u\|_2^2 - \alpha \|u\|_2^2 + B\|u\|_1^2}{\|u\|_{2^\star}^2} \tag{12.1.1}$$

For $\delta > 0$ small, let u_δ be the function of $H_{0,1}^2(\mathcal{B})$ defined by

$$u_\delta(x) = \left(\delta + |x|^2\right)^{1-\frac{n}{2}} - (\delta + 1)^{1-\frac{n}{2}}$$

Taking the u_δ's as test functions, it is easily seen that for any $B > 0$, $\lambda_B < \frac{1}{K_n}$. On such developments, we refer to Druet-Hebey-Vaugon [19] and Hebey [25]. Now we claim that the following holds:

(i) λ_B is continuous in B and increasing in B ,
(ii) $\lambda_B \to \frac{1}{K_n}$ as $B \to +\infty$.

Point (i) is easy to get. Just note that if $B_2 = B_1 + \eta$, $\eta \geq 0$, then

$$\lambda_{B_1} \leq \lambda_{B_2} \leq \lambda_{B_1} + \eta V_{\mathcal{B}}^{\frac{2(2^\star-1)}{2^\star}}$$

where $V_{\mathcal{B}}$ is the volume of $\mathcal{B}$. Concerning point (ii), let $B > 0$ be given. Since K_n^{-1} is the minimum energy for blow up, and $\lambda_B < K_n^{-1}$, classical variational methods lead to the existence of a minimizer for λ_B. In particular, we refer to section 7, there exists $u_B \in C^{1,\delta}(\overline{\mathcal{B}})$, $\delta \in (0,1)$, $u_B \geq 0$ in $\mathcal{B}$ and $u_B = 0$ on $\partial\mathcal{B}$, such that

$$\Delta u_B - \alpha u_B + B\|u_B\|_1 \Sigma_B = \lambda_B u_B^{2^\star-1} \tag{12.1.2}$$

and $\int_{\mathcal{B}} u_B^{2^\star} dx = 1$, where $\Sigma_B \in L^\infty(\mathcal{B})$ is such that $0 \leq \Sigma_B \leq 1$ and $\Sigma_B u_B = u_B$. Multiplying (12.1.2) by u_B and integrating over $\mathcal{B}$, we get that $B\|u_B\|_1^2 \leq \lambda_B$. As a consequence, $u_B \to 0$ in $L^1(\mathcal{B})$ as $B \to +\infty$. This implies that blow up occurs as $B \to +\infty$, and thus that $\lambda_B \to K_n^{-1}$ as $B \to +\infty$. Points (i) and (ii) above are proved.

Given $\varepsilon > 0$ small, we let $B_\varepsilon > 0$ be such that

$$\lambda_{B_\varepsilon} = \frac{1-\varepsilon}{K_n} \tag{12.1.3}$$

The goal in this section is to describe the asymptotic behavior of B_ε in terms of ε as $\varepsilon \to 0$. More precisely, we want to prove that

$$\lim_{\varepsilon\to 0} \frac{B_\varepsilon}{|\ln\varepsilon|^3} = \frac{3}{32\omega_3}\alpha^3 \tag{12.1.4}$$

when $n = 4$, and

$$\lim_{\varepsilon\to 0} B_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} = C_n \left(\frac{4(n-1)}{n-2}\alpha\right)^{\frac{n+2}{2}} \tag{12.1.5}$$

when $n \geq 5$, where

$$C_n = \frac{2n(n+2)\omega_n^{2+\frac{4}{n}}}{\omega_{n-1}^{\frac{2n}{n-2}}\left(4^{n-3}n(n-2)(n-4)\right)^{\frac{n+2}{n-2}}}$$

By standard symmetrization arguments, based on the co-area formula, functions in (12.1.1) can be assumed to be radially symmetrical and decreasing. As above, we then get the existence of a decreasing radially symmetrical function $u_\varepsilon \in C^{1,\delta}(\overline{\mathcal{B}})$, $\delta \in (0,1)$, $u_\varepsilon \geq 0$ in $\mathcal{B}$ and $u_\varepsilon = 0$ on $\partial\mathcal{B}$, such that

$$\Delta u_\varepsilon - \alpha u_\varepsilon + B_\varepsilon\|u_\varepsilon\|_1 \Sigma_\varepsilon = \frac{1-\varepsilon}{K_n} u_\varepsilon^{2^\star-1} \tag{12.1.6}$$

and $\int_{\mathcal{B}} u_\varepsilon^{2^\star} dx = 1$. There, see section 7, $\Sigma_\varepsilon \in L^\infty(\mathcal{B})$ is such that $\Sigma_\varepsilon = 1$ if $u_\varepsilon > 0$, and $\Sigma_\varepsilon = 0$ if $u_\varepsilon = 0$. In particular, there exists $r_\varepsilon \in (0,1]$ such that $\mathrm{Supp} u_\varepsilon = \mathcal{B}_0(r_\varepsilon)$, where $\mathcal{B}_0(r_\varepsilon)$ is the Euclidean ball of center 0 and radius r_ε. Then,

$$\Sigma_\varepsilon = 1 \text{ in } \mathcal{B}_0(r_\varepsilon) \text{ and } \Sigma_\varepsilon = 0 \text{ in } \overline{\mathcal{B}}\backslash\mathcal{B}_0(r_\varepsilon) \tag{12.1.7}$$

and, as a consequence, u_ε is smooth around 0. Since for any B, $\lambda_B < K_n^{-1}$, we have that $B_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Independently, multiplying (12.1.6) by u_ε and integrating over $\mathcal{B}$, we see that $B_\varepsilon \|u_\varepsilon\|_1^2$ is bounded as $\varepsilon \to 0$. By the preceding remark, this implies that $\|u_\varepsilon\|_1 \to 0$ as $\varepsilon \to 0$. Thus blow up must occur, and we are lead to the study of the asymptotic behavior of the u_ε's. A somehow similar problem was studied in Adimurthi-Pacella-Yadava [1]. This paper was concerned with the standard Euclidean sharp Sobolev inequality with Neumann boundary condition. As a starting point in the proof of (12.1.4) and (12.1.5) we prove weak estimates on the u_ε's.

12.1.1 Weak Estimates

We let $\mu_\varepsilon > 0$ be given by

$$u_\varepsilon(0) = \|u_\varepsilon\|_\infty = \mu_\varepsilon^{1-\frac{n}{2}} \tag{12.1.8}$$

Then, $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. Since $\Delta u_\varepsilon(0) \geq 0$ and $\Sigma_\varepsilon(0) = 1$, (12.1.6) gives that

$$B_\varepsilon \|u_\varepsilon\|_1 \leq \alpha \mu_\varepsilon^{1-\frac{n}{2}} + \frac{1-\varepsilon}{K_n} \mu_\varepsilon^{-1-\frac{n}{2}}$$

Thus,

$$B_\varepsilon \|u_\varepsilon\|_1 \mu_\varepsilon^{1+\frac{n}{2}} \leq \frac{2}{K_n} \tag{12.1.9}$$

for $\varepsilon > 0$ sufficiently small. Now, we let $\tilde{u}_\varepsilon$ be defined by

$$\tilde{u}_\varepsilon(x) = \mu_\varepsilon^{\frac{n}{2}-1} u_\varepsilon(\mu_\varepsilon x)$$

It is easily seen that

$$\Delta \tilde{u}_\varepsilon - \alpha \mu_\varepsilon^2 \tilde{u}_\varepsilon + B_\varepsilon \|u_\varepsilon\|_1 \mu_\varepsilon^{\frac{n}{2}+1} \tilde{\Sigma}_\varepsilon = \frac{1-\varepsilon}{K_n} \tilde{u}_\varepsilon^{2^\star-1} \tag{12.1.10}$$

in $\mathcal{B}_0(\mu_\varepsilon^{-1})$, where $\tilde{\Sigma}_\varepsilon(x) = \Sigma_\varepsilon(\mu_\varepsilon x)$. Noting that $\tilde{u}_\varepsilon \leq 1$, and thanks to (12.1.9), we get by standard elliptic theory that the $\tilde{u}_\varepsilon$'s are equicontinuous on any compact subset of $\mathbb{R}^n$. By Ascoli's theorem we then get that there exists $u_0 \in C^0(\mathbb{R}^n)$ such that, after passing to a subsequence,

$$\tilde{u}_\varepsilon \to u_0 \text{ in } C^0_{loc}(\mathbb{R}^n) \tag{12.1.11}$$

Clearly, $u_0(0) = 1$, and we have that $u_0 \in D_1^2(\mathbb{R}^n)$, where $D_1^2(\mathbb{R}^n)$ is the homogeneous Euclidean Sobolev space. Up to a subsequence, we define Σ_0 by

$$\Sigma_0(x) = \lim_{\varepsilon \to 0} B_\varepsilon \|u_\varepsilon\|_1 \mu_\varepsilon^{1+\frac{n}{2}} \Sigma_\varepsilon(\mu_\varepsilon x)$$

Assuming that $\frac{r_\varepsilon}{\mu_\varepsilon} \to R$ as $\varepsilon \to 0$, and that $B_\varepsilon \|u_\varepsilon\|_1 \mu_\varepsilon^{1+\frac{n}{2}} \to A$ as $\varepsilon \to 0$, we then have that $\Sigma_0 = 0$ if $R = 0$, $\Sigma_0 = A$ if $R = +\infty$, and $\Sigma_0 = A \mathbb{1}_{\mathcal{B}_0(R)}$ if $R \in (0, +\infty)$, where $\mathbb{1}_X$ stands for the characteristic function of a subset X of $\mathbb{R}^n$. It is easily seen that u_0 is a solution in $\mathbb{R}^n$ of the equation

$$\Delta u_0 + \Sigma_0 = \frac{1}{K_n} u_0^{2^\star-1} \tag{12.1.12}$$

We claim that this implies that $\Sigma_0 = 0$. When $R \in (0, +\infty)$, such a claim easily follows from the Pohozaev identity [33]. Note that in this case, u_0 is compactly supported in $\mathcal{B}_0(R)$. Let us assume now that $R = +\infty$ and $A > 0$. Multiplying (12.1.10) by $\tilde{u}_\varepsilon$ and integrating, we easily get that $u_0 \in L^1(I\!R^n)$. By standard regularity results, we also have that u_0 is $C^{2,k}$, $k \in (0,1)$. We let η be a smooth cut-off function such that $0 \leq \eta \leq 1$, $\eta = 1$ if $|x| \leq 1$, and $\eta = 0$ if $|x| \geq 2$. For $r > 0$, we let also η_r be given by

$$\eta_r(x) = \eta\left(\frac{x}{r}\right)$$

The Pohozaev identity [33], applied to $\eta_r u_0$, gives that

$$2\int_{I\!R^n} \left(\nabla(\eta_r u_0), x\right) \Delta(\eta_r u_0) dx + (n-2)\int_{I\!R^n} \eta_r u_0 \Delta(\eta_r u_0) dx \leq 0 \tag{12.1.13}$$

Moreover, $\left(\nabla\eta_r\right)(x) = \frac{1}{r}\nabla\eta\left(\frac{x}{r}\right)$ and $\left(\Delta\eta_r\right)(x) = \frac{1}{r^2}\Delta\eta\left(\frac{x}{r}\right)$. Integrating by parts, using the Lebesgue dominated convergence theorem, and thanks to (12.1.12),

$$\int_{I\!R^n} \left(\nabla(\eta_r u_0), x\right) \Delta(\eta_r u_0) dx = -\frac{n-2}{2K_n}\int_{I\!R^n} \eta_r^2 u_0^{2^\star} dx + nA\int_{I\!R^n} \eta_r^2 u_0 dx + o(1)$$
$$\int_{I\!R^n} \eta_r u_0 \Delta(\eta_r u_0) dx = \frac{1}{K_n}\int_{I\!R^n} \eta_r^2 u_0^{2^\star} dx - A\int_{I\!R^n} \eta_r^2 u_0 dx + o(1)$$

where $o(1) \to 0$ as $r \to +\infty$. Coming back to (12.1.13), it follows that

$$A\int_{I\!R^n} \eta_r^2 u_0 dx \leq o(1)$$

and, passing to the limit as $r \to +\infty$, we get a contradiction. Thus, $A = 0$ if $R = +\infty$, and this proves the above claim. In particular, u_0 is a solution of the equation

$$\Delta u_0 = \frac{1}{K_n} u_0^{2^\star - 1}$$

By Caffarelli-Gidas-Spruck [8], and also Obata [32], it follows that

$$u_0(x) = \left(\frac{1}{1 + \frac{\omega_n^{2/n}}{4}|x|^2}\right)^{\frac{n-2}{2}}$$

Noting that $\mathrm{Supp}\tilde{u}_\varepsilon \subset \mathcal{B}_0(\frac{r_\varepsilon}{\mu_\varepsilon})$, we get that $\frac{r_\varepsilon}{\mu_\varepsilon} \to +\infty$ as $\varepsilon \to 0$. Another consequence is that

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{\mathcal{B}_0(R\mu_\varepsilon)} u_\varepsilon^{2^\star} dx = 1 \tag{12.1.14}$$

We claim that (12.1.14) implies in turn that the two following estimates hold. On the one hand, there exists $C > 0$ such that for any $\varepsilon > 0$ and any $x \in \mathcal{B}$,

$$|x|^{\frac{n}{2}-1} u_\varepsilon(x) \leq C \tag{12.1.15}$$

On the other hand,

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\sup_{\mathcal{B}\backslash\mathcal{B}_0(R\mu_\varepsilon)}|x|^{\frac{n}{2}-1}u_\varepsilon(x)=0 \tag{12.1.16}$$

We prove (12.1.15). Let v_ε be defined by

$$v_\varepsilon(x)=|x|^{\frac{n}{2}-1}u_\varepsilon(x)$$

We assume by contradiction that for some subsequence, $\|v_\varepsilon\|_\infty\to+\infty$ as $\varepsilon\to 0$. Let x_ε be a point in $\mathcal{B}$ where v_ε is maximum. A straitghforward consequence of (12.1.14) is that for $x\neq 0$, and $\delta>0$ sufficiently small,

$$\int_{\mathcal{B}_x(\delta)}u_\varepsilon^{2^\star}dx\to 0$$

as $\varepsilon\to 0$. Let $x\in\mathcal{B}$, $x\neq 0$, and η be a smooth cut-off function around x. Multiplying (12.1.6) by $\eta^2u_\varepsilon^k$, $k\geq 1$, and integrating over $\mathcal{B}$, it is easily seen, see for instance section 8, that for $\delta>0$ sufficiently small, the u_ε's are bounded in $L^{(2^\star)^2/2}\left(\mathcal{B}_x(\delta)\right)$. Since $(2^\star)^2/2>2^\star$, it follows from the De Giorgi-Nash-Moser iterative scheme and (12.1.6) that

$$u_\varepsilon\to 0\ \text{ in } C^0_{loc}(\mathcal{B}\backslash\{0\}) \tag{12.1.17}$$

as $\varepsilon\to 0$. In particular, (12.1.17) implies that $x_\varepsilon\to 0$ as $\varepsilon\to 0$. Since $u_\varepsilon(x_\varepsilon)\leq u_\varepsilon(0)$ and $\|v_\varepsilon\|_\infty\to+\infty$, we also have that

$$\frac{|x_\varepsilon|}{\mu_\varepsilon}\to+\infty \tag{12.1.18}$$

as $\varepsilon\to 0$, and that $u_\varepsilon(x_\varepsilon)\to+\infty$ as $\varepsilon\to 0$. We set

$$\Omega_\varepsilon=u_\varepsilon(x_\varepsilon)^{\frac{2}{n-2}}\mathcal{B}_{-x_\varepsilon}(1)$$

and for $x\in\Omega_\varepsilon$, we set

$$\tilde{v}_\varepsilon(x)=u_\varepsilon(x_\varepsilon)^{-1}u_\varepsilon\left(x_\varepsilon+u_\varepsilon(x_\varepsilon)^{-\frac{2}{n-2}}x\right)$$

It is easily seen that for $\varepsilon>0$ small, and all $x\in\mathcal{B}_0(2)$,

$$\left|x_\varepsilon+u_\varepsilon(x_\varepsilon)^{-\frac{2}{n-2}}x\right|\geq\frac{1}{2}\left|x_\varepsilon\right| \tag{12.1.19}$$

Then, for all $x\in\mathcal{B}_0(2)$,

$$\begin{array}{rcl}\tilde{v}_\varepsilon(x) & \leq & 2^{\frac{n}{2}-1}|x_\varepsilon|^{1-\frac{n}{2}}u_\varepsilon(x_\varepsilon)^{-1}v_\varepsilon\left(x_\varepsilon+u_\varepsilon(x_\varepsilon)^{-\frac{2}{n-2}}x\right)\\ & \leq & 2^{\frac{n}{2}-1}|x_\varepsilon|^{1-\frac{n}{2}}u_\varepsilon(x_\varepsilon)^{-1}v_\varepsilon(x_\varepsilon)\end{array}$$

so that for $\varepsilon>0$ small,

$$\sup_{x\in\mathcal{B}_0(2)}\tilde{v}_\varepsilon(x)\leq 2^{\frac{n}{2}-1} \tag{12.1.20}$$

Let $R>0$ be given. By (12.1.18) and (12.1.19),

$$\mathcal{B}_{x_\varepsilon}\left(2u_\varepsilon(x_\varepsilon)^{-\frac{2}{n-2}}\right)\cap\mathcal{B}_0\left(R\mu_\varepsilon\right)=\emptyset \tag{12.1.21}$$

for $\varepsilon > 0$ small. Noting that

$$\int_{\mathcal{B}_0(2)} \tilde{v}_\varepsilon^{2^\star} dx = \int_{\mathcal{B}_{x_\varepsilon}(2u_\varepsilon(x_\varepsilon)^{-\frac{2}{n-2}})} u_\varepsilon^{2^\star} dx$$

it follows from (12.1.14) and (12.1.21) that

$$\int_{\mathcal{B}_0(2)} \tilde{v}_\varepsilon^{2^\star} dx \to 0 \tag{12.1.22}$$

as $\varepsilon \to 0$. As is easily checked,

$$\Delta \tilde{v}_\varepsilon - \alpha u_\varepsilon(x_\varepsilon)^{-4/(n-2)} \tilde{v}_\varepsilon \leq \frac{1-\varepsilon}{K_n} \tilde{v}_\varepsilon^{2^\star - 1}$$

The De Giorgi-Nash-Moser iterative scheme, (12.1.20) and (12.1.22) then give that

$$\sup_{x \in \mathcal{B}_0(1)} \tilde{v}_\varepsilon(x) \to 0$$

as $\varepsilon \to 0$. Since $\tilde{v}_\varepsilon(0) = 1$, we get a contradiction. This proves (12.1.15). The proof of (12.1.16), that we omit here, goes in the same way. On such a claim, see Druet [13], or subsection 12.1.3 below.

Going on with the asymptotic study of the u_ε's, we claim that $r_\varepsilon \to 0$ as $\varepsilon \to 0$. We let $\delta > 0$ and $\eta \in C_0^\infty(\mathcal{B})$ be such that $\eta = 0$ in $\mathcal{B}_0(\frac{\delta}{2})$, $\eta = 1$ in $\mathcal{B}_0(\frac{1}{2}) \backslash \mathcal{B}_0(\delta)$. Multiplying (12.1.6) by η and integrating over $\mathcal{B}$, we get with (12.1.17) that

$$\begin{aligned} B_\varepsilon \|u_\varepsilon\|_1 \int_{\mathcal{B}} \eta \Sigma_\varepsilon dx &= \frac{1-\varepsilon}{K_n} \int_{\mathcal{B}} \eta u_\varepsilon^{2^\star - 1} dx + \alpha \int_{\mathcal{B}} \eta u_\varepsilon dx - \int_{\mathcal{B}} (\Delta \eta) u_\varepsilon dx \\ &= O\left(\|u_\varepsilon\|_1\right) \end{aligned}$$

Since $B_\varepsilon \to +\infty$ as $\varepsilon \to 0$, it follows that $\int_{\mathcal{B}} \eta \Sigma_\varepsilon dx \to 0$ as $\varepsilon \to 0$. In particular,

$$\int_{\mathcal{B}_0(\frac{1}{2}) \backslash \mathcal{B}_0(\delta)} \Sigma_\varepsilon dx \to 0$$

as $\varepsilon \to 0$, and since this holds for any $\delta > 0$, we get that $r_\varepsilon \to 0$ as $\varepsilon \to 0$. Now we prove stronger estimates than (12.1.15) and (12.1.16).

12.1.2 Strong Estimates

We define L_ε by

$$L_\varepsilon u = \Delta u - \alpha u - \frac{1-\varepsilon}{K_n} u_\varepsilon^{2^\star - 2} u$$

Letting $\delta > 0$ sufficiently small so that $\Delta - \alpha$ is coercive on $\mathcal{B}_0(\delta)$, we claim first that L_ε satisfies the maximum principle on $\mathcal{B}_0(\delta) \backslash \mathcal{B}_0(R\mu_\varepsilon)$ for $R > 0$ large and $\varepsilon > 0$ small. Let indeed $z \in C^1\left(\overline{\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)}\right)$ be such that $z \geq 0$ on $\partial\left(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)\right)$ and $L_\varepsilon z \geq 0$. Set $z^- = \max(0, -z)$. Then,

$$\begin{aligned} 0 &\leq \int_{\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)} z^- L_\varepsilon z dx \\ &= -\|\nabla z^-\|^2_{L^2(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} + \alpha\|z^-\|^2_{L^2(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} \\ &\quad + \frac{1-\varepsilon}{K_n}\int_{\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)} u_\varepsilon^{2^\star-2}(z^-)^2 dx \end{aligned}$$

while, thanks to Hölder's inequality,

$$\int_{\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)} u_\varepsilon^{2^\star-2}(z^-)^2 dx \leq \|u_\varepsilon\|^{2^\star-2}_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))}\|z^-\|^2_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))}$$

Thus,

$$\begin{aligned} 0 \leq &-\|\nabla z^-\|^2_{L^2(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} + \alpha\|z^-\|^2_{L^2(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} \\ &+ \frac{1-\varepsilon}{K_n}\|u_\varepsilon\|^{2^\star-2}_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))}\|z^-\|^2_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} \end{aligned} \tag{12.1.23}$$

By (12.1.14),

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} = 0$$

It follows that for any $A > 0$, there exist $\varepsilon_A > 0$ and $R_A > 0$ such that for $R \geq R_A$ and $\varepsilon \in (0, \varepsilon_A)$,

$$\|u_\varepsilon\|_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} \leq A$$

Let $B > 0$, given by the coercivity of $L = \Delta - \alpha$ on $\mathcal{B}_0(\delta)$, be such that

$$B\|z^-\|^2_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} \leq \|\nabla z^-\|^2_{L^2(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} - \alpha\|z^-\|^2_{L^2(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))}$$

Coming back to (12.1.23), we have

$$0 \leq \|z^-\|^2_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))}\left(\frac{1-\varepsilon}{K_n}A^{2^\star-2} - B\right)$$

Choosing $A > 0$ small, this implies that $z^- = 0$. The claim is proved. From now on, we let $c > 0$ be such that $\tilde{L} = \Delta - (\alpha + c)$ is coercive on $\mathcal{B}_0(\delta)$. We let also G be the Green function of $\tilde{L}$ in $\mathcal{B}_0(\delta)$ with zero Dirichlet boundary condition, and set $H(x) = G(0, x)$. We fix $\nu > 0$ small, sufficiently small so that $(1-\nu)c - \nu\alpha > 0$. Then, in $\mathcal{B}_0(\delta)\backslash\{0\}$,

$$\frac{L_\varepsilon H^{1-\nu}}{H^{1-\nu}} = \nu(1-\nu)\frac{|\nabla H|^2}{H^2} + \alpha_0 - \frac{1-\varepsilon}{K_n}u_\varepsilon^{2^\star-2} \tag{12.1.24}$$

where $\alpha_0 > 0$ is given by $\alpha_0 = (1-\nu)c - \nu\alpha$. An easy property of the Green function is that there exists $C_0 > 0$ and $\rho_0 > 0$ such that for $|x| \leq \rho_0$,

$$\frac{|\nabla H|^2}{H^2} \geq \frac{C_0}{|x|^2}$$

Thanks to (12.1.16), for $R > 0$ large and $\varepsilon > 0$ small,

$$\frac{\nu(1-\nu)C_0}{|x|^2} \geq \frac{1-\varepsilon}{K_n} u_\varepsilon^{2^\star-2}$$

in $\mathcal{B}_0(\delta) \backslash \mathcal{B}_0(R\mu_\varepsilon)$. Coming back to (12.1.24), it follows that $L_\varepsilon H^{1-\nu} \geq 0$ in the annulus $\mathcal{B}_0(\rho_0) \backslash \mathcal{B}_0(R\mu_\varepsilon)$. By (12.1.17), since $\alpha_0 > 0$, we also have that for $\varepsilon > 0$ small, $L_\varepsilon H^{1-\nu} \geq 0$ outside $\mathcal{B}_0(\rho_0)$. Hence, $L_\varepsilon H^{1-\nu} \geq 0$ in $\mathcal{B}_0(\delta) \backslash \mathcal{B}_0(R\mu_\varepsilon)$ provided that $R > 0$ is large and $\varepsilon > 0$ is small. We fix $R > 0$ large. By (12.1.15), there exists $C_1 > 0$ such that

$$u_\varepsilon \leq C_1 \mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)} |x|^{(2-n)(1-\nu)}$$

on $\partial\mathcal{B}_0(R\mu_\varepsilon)$. We also have that there exists $C_2 > 0$ such that $H \geq C_2|x|^{2-n}$ around 0, and that there exists $C_3 > 0$ such that $H \leq C_3|x|^{2-n}$. Then, since $L_\varepsilon u_\varepsilon = 0$ and $u_\varepsilon = 0$ on $\partial\mathcal{B}_0(\delta)$, we get that there exists $C_4 > 0$ such that

$$\begin{aligned} &L_\varepsilon \left(C_4 \mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)} H^{1-\nu} - u_\varepsilon \right) \geq 0 \text{ in } \mathcal{B}_0(\delta) \backslash \mathcal{B}_0(R\mu_\varepsilon) \text{ , and} \\ &C_4 \mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)} H^{1-\nu} \geq u_\varepsilon \text{ on } \partial \left(\mathcal{B}_0(\delta) \backslash \mathcal{B}_0(R\mu_\varepsilon) \right) \end{aligned}$$

By the maximum principle, it follows that

$$u_\varepsilon \leq C_4 \mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)} H^{1-\nu}$$

in $\mathcal{B}_0(\delta) \backslash \mathcal{B}_0(R\mu_\varepsilon)$, and then that

$$u_\varepsilon \leq C_5 \mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)} |x|^{(1-\nu)(2-n)}$$

in $\mathcal{B}_0(\delta) \backslash \mathcal{B}_0(R\mu_\varepsilon)$ for some $C_5 > 0$. It is clear that this inequality holds also in $\mathcal{B}_0(R\mu_\varepsilon)$, up to changing C_5. As a consequence, we proved that for $\nu > 0$ small, there exists $C_6 > 0$, such that for $\varepsilon > 0$ small,

$$u_\varepsilon \leq C_6 \mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)} |x|^{(1-\nu)(2-n)} \tag{12.1.25}$$

in $\mathcal{B}_0(\delta)$, and thus, also in $\mathcal{B}$. Pushing further the analysis, we let now (y_ε) be a sequence of points in $\mathcal{B}_0(\frac{\delta_0}{2})$, and let $\tilde{G}$ be the Green function of $L = \Delta - \alpha$ in $\mathcal{B}_0(\delta_0)$ with zero Dirichlet boundary condition, where $\delta_0 > 0$ is such that $L = \Delta - \alpha$ is coercive on $\mathcal{B}_0(\delta_0)$. Thanks to (12.1.6), and since $r_\varepsilon \to 0$ as $\varepsilon \to 0$,

$$u_\varepsilon(y_\varepsilon) \leq \frac{1}{K_n} \int_{\mathcal{B}_0(\delta_0)} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star-1}(x) dx \tag{12.1.26}$$

We set

$$\Phi_\varepsilon = u_\varepsilon(y_\varepsilon) \mu_\varepsilon^{1-\frac{n}{2}} |y_\varepsilon|^{n-2}$$

and distinguish three cases.

Case 1: we assume that $\frac{|y_\varepsilon|}{\mu_\varepsilon} \to R$ as $\varepsilon \to 0$, $R \in [0, +\infty)$. Then, thanks to (12.1.15), (Φ_ε) is bounded.

Case 2: we assume that $y_\varepsilon \to y_0$ as $\varepsilon \to 0$, where $y_0 \neq 0$, and let $\delta > 0$ be such that $2\delta \leq |y_0|$. Then,

$$\begin{aligned}
&\int_{\mathcal{B}_0(\delta_0)} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star-1}(x)dx \\
&\leq \int_{\mathcal{B}_0(\delta)} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star-1}(x)dx + \int_{\mathcal{B}_0(\delta_0)\backslash\mathcal{B}_0(\delta)} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star-1}(x)dx \\
&\leq C\int_{\mathcal{B}_0(\delta)} u_\varepsilon^{2^\star-1}dx + C\int_{\mathcal{B}_0(\delta_0)\backslash\mathcal{B}_0(\delta)} \frac{1}{|y_\varepsilon - x|^{n-2}} u_\varepsilon^{2^\star-1}dx
\end{aligned}$$

where $C > 0$ is independent of ε. By (12.1.25), with $(n+2)\nu < 2$,

$$\int_{\mathcal{B}_0(\delta_0)\backslash\mathcal{B}_0(\delta)} \frac{1}{|y_\varepsilon - x|^{n-2}} u_\varepsilon^{2^\star-1}dx = o\left(\mu_\varepsilon^{\frac{n}{2}-1}\right)$$

Independently,

$$\int_{\mathcal{B}_0(\delta)} u_\varepsilon^{2^\star-1}dx = \int_{\mathcal{B}_0(\mu_\varepsilon)} u_\varepsilon^{2^\star-1}dx + \int_{\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(\mu_\varepsilon)} u_\varepsilon^{2^\star-1}dx$$

By (12.1.11),

$$\int_{\mathcal{B}_0(\mu_\varepsilon)} u_\varepsilon^{2^\star-1}dx = O\left(\mu_\varepsilon^{\frac{n}{2}-1}\right)$$

while by (12.1.25) where $\nu > 0$ is chosen sufficiently small such that $(n+2)\nu < 2$,

$$\int_{\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(\mu_\varepsilon)} u_\varepsilon^{2^\star-1}dx = O\left(\mu_\varepsilon^{\frac{n}{2}-1}\right)$$

By (12.1.26), this implies that (Φ_ε) is bounded.

Case 3: we assume that $\frac{|y_\varepsilon|}{\mu_\varepsilon} \to +\infty$ and that $|y_\varepsilon| \to 0$ as $\varepsilon \to 0$. Then, by (12.1.25),

$$\begin{aligned}
&\int_{\mathcal{B}_0(\delta_0)} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star-1}(x)dx \\
&\leq \int_{\mathcal{B}_{y_\varepsilon}(\frac{|y_\varepsilon|}{2})} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star-1}(x)dx + \int_{\mathcal{B}_0(\delta_0)\backslash\mathcal{B}_{y_\varepsilon}(\frac{|y_\varepsilon|}{2})} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star-1}(x)dx \\
&\leq C\mu_\varepsilon^{\frac{n+2}{2}(1-2\nu)}|y_\varepsilon|^{(\nu-1)(n+2)} \int_{\mathcal{B}_{y_\varepsilon}(\frac{|y_\varepsilon|}{2})} \frac{1}{|y_\varepsilon - x|^{n-2}}dx \\
&\quad +C\frac{1}{|y_\varepsilon|^{n-2}} \int_{\mathcal{B}_0(\delta_0)} u_\varepsilon^{2^\star-1}dx \\
&\leq C\mu_\varepsilon^{\frac{n+2}{2}(1-2\nu)}|y_\varepsilon|^{(\nu-1)(n+2)+2} + C\frac{1}{|y_\varepsilon|^{n-2}}\mu_\varepsilon^{\frac{n}{2}-1}
\end{aligned}$$

where $C > 0$ does not depend on ε. Thanks to (12.1.26) we then get that

$$|y_\varepsilon|^{n-2}\mu_\varepsilon^{1-\frac{n}{2}}u_\varepsilon(y_\varepsilon) \leq C\left(\frac{\mu_\varepsilon}{|y_\varepsilon|}\right)^{2-(n+2)\nu} + C$$

and since $\frac{|y_\varepsilon|}{\mu_\varepsilon} \to +\infty$ as $\varepsilon \to 0$, we get that (Φ_ε) is bounded.

Summarizing cases 1 to 3, for any sequence (y_ε) in $\mathcal{B}_0(\frac{\delta_0}{2})$, there exists $C>0$ such that

$$\mu_\varepsilon^{1-\frac{n}{2}}|y_\varepsilon|^{n-2}u_\varepsilon(y_\varepsilon)\le C$$

Since the u_ε's are radially decreasing, this implies that there exists $C>0$ such that for any $x\in\mathcal{B}$ and any $\varepsilon>0$,

$$\mu_\varepsilon^{1-\frac{n}{2}}|x|^{n-2}u_\varepsilon(x)\le C \tag{12.1.27}$$

An equivalent formulation of (12.1.27) is that for any $x\in\mathcal{B}$ and any $\varepsilon>0$,

$$u_\varepsilon(x)\le C\mu_\varepsilon^{1-\frac{n}{2}}\left(\frac{1}{1+\frac{\omega_n^{2/n}}{4\mu_\varepsilon^2}|x|^2}\right)^{\frac{n-2}{2}} \tag{12.1.28}$$

where $C>0$ is independent of x and ε.

Going on with the proof of (12.1.4) and (12.1.5), the goal of the following subsection is to estimate r_ε in terms of μ_ε. We start with the case $n\ge 5$.

12.1.3 Estimating r_ε with respect to μ_ε when $n\ge 5$

As already mentioned, we want to estimate r_ε in terms of μ_ε. For that purpose, we define the function $\hat{u}_\varepsilon$ by

$$\hat{u}_\varepsilon(x)=r_\varepsilon^{\frac{n}{2}-1}u_\varepsilon(r_\varepsilon x) \tag{12.1.29}$$

It is easily seen that $\hat{u}_\varepsilon>0$ in $\mathcal{B}$, $\hat{u}_\varepsilon=0$ on $\partial\mathcal{B}$,

$$\Delta\hat{u}_\varepsilon-\alpha r_\varepsilon^2\hat{u}_\varepsilon+B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}=\frac{1-\varepsilon}{K_n}\hat{u}_\varepsilon^{2^\star-1} \tag{12.1.30}$$

in $\mathcal{B}$, and

$$\int_{\mathcal{B}}\hat{u}_\varepsilon^{2^\star}dx=1 \tag{12.1.31}$$

Moreover, if we set $\hat{\mu}_\varepsilon=\mu_\varepsilon/r_\varepsilon$, then

$$\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)\to\left(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\right)^{\frac{n-2}{2}} \tag{12.1.32}$$

in $C^0_{loc}(I\!R^n)$, and, thanks to (12.1.27),

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}|x|^{n-2}\hat{u}_\varepsilon(x)\le C \tag{12.1.33}$$

for any $x\in\mathcal{B}$. By (12.1.32), $\hat{\mu}_\varepsilon\to 0$ as $\varepsilon\to 0$. As another remark, since u_ε is C^1 in $\mathcal{B}$, we have that

$$\partial_\nu\hat{u}_\varepsilon=0\ \text{ on }\partial\mathcal{B} \tag{12.1.34}$$

Multiplying (12.1.30) by $\hat{\mu}_\varepsilon^{1-\frac{n}{2}}$ and integrating we get that

$$-\alpha r_\varepsilon^2\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon dx+B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}|\mathcal{B}|=\frac{1-\varepsilon}{K_n}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon^{2^\star-1}dx$$

where $|\mathcal{B}|$ is the volume of $\mathcal{B}$. Since

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon(x)^{2^\star-1}dx=\int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)\right)^{2^\star-1}dx$$

we get with (12.1.32) and (12.1.33) that

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon(x)^{2^\star-1}dx\to\int_{I\!R^n}\left(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\right)^{\frac{n+2}{2}}dx$$

as $\varepsilon\to 0$. Independently, since $r_\varepsilon\to 0$ as $\varepsilon\to 0$, (12.1.33) gives that

$$r_\varepsilon^2\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon dx\to 0$$

as $\varepsilon\to 0$. Noting that

$$\frac{1}{K_n}\int_{I\!R^n}\left(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\right)^{\frac{n+2}{2}}dx=(n-2)2^{n-2}\omega_n^{\frac{2}{n}-1}\omega_{n-1}$$

it follows that

$$B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\to A_n \tag{12.1.35}$$

as $\varepsilon\to 0$, where

$$A_n=n(n-2)2^{n-2}\omega_n^{\frac{2}{n}-1} \tag{12.1.36}$$

We have that

$$\Delta(\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon)-\alpha r_\varepsilon^2\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon+B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}=\frac{1-\varepsilon}{K_n}\hat{\mu}_\varepsilon^2\left(\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon\right)^{2^\star-1}$$

and the coefficients in this equation are bounded thanks to (12.1.35). Since the sequence $(\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon)$ is bounded in any compact subset of $\overline{\mathcal{B}}\backslash\{0\}$, we get by standard elliptic theory that

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon\to\Phi\ \text{ in } C^1_{loc}\left(\overline{\mathcal{B}}\backslash\{0\}\right) \tag{12.1.37}$$

where Φ is a solution of

$$\Delta\Phi+A_n=0 \tag{12.1.38}$$

in $\mathcal{B}\backslash\{0\}$. Clearly, Φ is radially symmetrical and decreasing in $\overline{\mathcal{B}}\backslash\{0\}$. Moreover,

$$\Phi=0\ \text{ and }\ \partial_\nu\Phi=0\ \text{ on }\partial\mathcal{B}$$

Integrating (12.1.38) on $\mathcal{B}\backslash\mathcal{B}_0(r)$, we then get that

$$\Phi(x)=\frac{A_n}{n(n-2)}\left(\frac{1}{|x|^{n-2}}-1\right)+\frac{A_n}{2n}\left(|x|^2-1\right) \tag{12.1.39}$$

Now we apply the Pohozaev identity to $\hat{u}_\varepsilon$ in $\mathcal{B}$. The Pohozaev identity [33] for $\hat{u}_\varepsilon$ in $\mathcal{B}$ states that

$$\begin{aligned}\int_{\partial\mathcal{B}} (x,\nu)\,(\partial_\nu \hat{u}_\varepsilon)^2 d\sigma + (n-2)\int_{\partial\mathcal{B}} \hat{u}_\varepsilon(\partial_\nu \hat{u}_\varepsilon) d\sigma \\ = -2\int_{\mathcal{B}} (\nabla \hat{u}_\varepsilon, x)\,\Delta\hat{u}_\varepsilon dx - (n-2)\int_{\mathcal{B}} \hat{u}_\varepsilon \Delta\hat{u}_\varepsilon dx\end{aligned}$$

where ν is the unit outer normal to $\partial\mathcal{B}$. Since $\hat{u}_\varepsilon = 0$ and $\partial_\nu \hat{u}_\varepsilon = 0$ on $\partial\mathcal{B}$, we get with (12.1.30) that

$$\alpha r_\varepsilon^2 \int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx = \frac{n+2}{2} B_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \int_{\mathcal{B}} \hat{u}_\varepsilon dx$$

By (12.1.33), (12.1.35), (12.1.37), and (12.1.39), this implies that

$$\frac{1}{\hat{\mu}_\varepsilon^{n-2}} r_\varepsilon^2 \int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx \to \frac{n+2}{2\alpha} A_n \int_{\mathcal{B}} \Phi dx = \frac{A_n^2}{4n\alpha}\omega_{n-1} \tag{12.1.40}$$

as $\varepsilon \to 0$. Independently,

$$\frac{1}{\hat{\mu}_\varepsilon^2}\int_{\mathcal{B}} \hat{u}_\varepsilon(x)^2 dx = \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} \left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1} \hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)\right)^2 dx$$

and by (12.1.32) and (12.1.33) we get that when $n \geq 5$,

$$\int_{\mathcal{B}} \hat{u}_\varepsilon(x)^2 dx = \left(\int_{\mathbb{R}^n} \left(1 + \frac{\omega_n^{2/n}}{4}|x|^2\right)^{2-n} dx\right) \hat{\mu}_\varepsilon^2 + o\left(\hat{\mu}_\varepsilon^2\right)$$

It is easily seen, see for instance Demengel and Hebey [11], that

$$\int_{\mathbb{R}^n} \left(1 + \frac{\omega_n^{2/n}}{4}|x|^2\right)^{2-n} dx = 2^{n-1}\frac{\omega_{n-1}}{\omega_n}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}-2)}{\Gamma(n-2)}$$

where Γ is the Euler function. Since

$$\Gamma(n) = 2^{n-1}\frac{\omega_{n-1}}{\omega_n}\Gamma(\frac{n}{2})^2$$

we get that when $n \geq 5$,

$$\int_{\mathcal{B}} \hat{u}_\varepsilon(x)^2 dx = \frac{4(n-1)}{n-4}\hat{\mu}_\varepsilon^2 + o\left(\hat{\mu}_\varepsilon^2\right) \tag{12.1.41}$$

Combining (12.1.40) and (12.1.41), it follows that

$$\lim_{\varepsilon\to 0} \frac{r_\varepsilon^2}{\hat{\mu}_\varepsilon^{n-4}} = \frac{(n-4)\omega_{n-1}A_n^2}{16n(n-1)\alpha} \tag{12.1.42}$$

when $n \geq 5$, where A_n is given by (12.1.36).

The goal of the following subsection is to estimate r_ε in terms of μ_ε in the limit case $n = 4$. For that purpose, a stronger estimate than (12.1.28) is needed.

12.1.4 Estimating r_ε with respect to μ_ε when $n=4$

We claim that when $n=4$,

$$\lim_{\varepsilon\to 0}|\ln\hat\mu_\varepsilon|r_\varepsilon^2=\frac{4}{\alpha}\tag{12.1.43}$$

In order to prove this claim, we let (y_ε) be a sequence of points in $\mathcal{B}$ such that $y_\varepsilon\to 0$ and $\frac{|y_\varepsilon|}{\hat\mu_\varepsilon}\to+\infty$ as $\varepsilon\to 0$. We let also $\hat v_\varepsilon$ be the function given by

$$\hat v_\varepsilon(x)=|y_\varepsilon|^{\frac{n}{2}-1}\hat u_\varepsilon(|y_\varepsilon|x)$$

Then,

$$\Delta\hat v_\varepsilon-\alpha r_\varepsilon^2|y_\varepsilon|^2\hat v_\varepsilon+B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}|y_\varepsilon|^{\frac{n}{2}+1}=\frac{1-\varepsilon}{K_n}\hat v_\varepsilon^{2^\star-1}$$

in $\mathcal{B}_0(\frac{1}{|y_\varepsilon|})$, and if

$$\hat w_\varepsilon=\left(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\right)^{1-\frac{n}{2}}\hat v_\varepsilon$$

we get that

$$\begin{aligned}&\Delta\hat w_\varepsilon-\alpha r_\varepsilon^2|y_\varepsilon|^2\hat w_\varepsilon+B_\varepsilon\|u_\varepsilon\|_1\left(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\right)^{1-\frac{n}{2}}r_\varepsilon^{\frac{n}{2}+1}|y_\varepsilon|^{\frac{n}{2}+1}\\&=\frac{1-\varepsilon}{K_n}\left(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\right)^2\hat w_\varepsilon^{2^\star-1}\end{aligned}\tag{12.1.44}$$

By (12.1.32) and (12.1.33),

$$\left(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\right)^{n-2}\hat w_\varepsilon\left(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}x\right)\to\left(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\right)^{\frac{n-2}{2}}\tag{12.1.45}$$

in $C^0_{loc}(I\!R^n)$, and

$$|x|^{n-2}\hat w_\varepsilon(x)\le C\tag{12.1.46}$$

Integrating (12.1.44) over $\mathcal{B}_0(\frac{1}{|y_\varepsilon|})$, we get that

$$\begin{aligned}&B_\varepsilon\|u_\varepsilon\|_1\left(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\right)^{1-\frac{n}{2}}r_\varepsilon^{\frac{n}{2}+1}|y_\varepsilon|^{\frac{n}{2}+1}\frac{\omega_{n-1}}{n|y_\varepsilon|^n}\\&=\alpha r_\varepsilon^2|y_\varepsilon|^2\int_{\mathcal{B}_0(\frac{1}{|y_\varepsilon|})}\hat w_\varepsilon(x)dx+\frac{1-\varepsilon}{K_n}\left(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\right)^2\int_{\mathcal{B}_0(\frac{1}{|y_\varepsilon|})}\hat w_\varepsilon(x)^{2^\star-1}dx\end{aligned}\tag{12.1.47}$$

By (12.1.46),

$$|y_\varepsilon|^2\int_{\mathcal{B}_0(\frac{1}{|y_\varepsilon|})}\hat w_\varepsilon(x)dx=|y_\varepsilon|^{2-n}\int_{\mathcal{B}}\hat w_\varepsilon\left(\frac{x}{|y_\varepsilon|}\right)dx\le C\int_{\mathcal{B}}\frac{1}{|x|^{n-2}}dx=C'$$

Independently,

$$\left(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\right)^2\int_{\mathcal{B}_0(\frac{1}{|y_\varepsilon|})}\hat w_\varepsilon(x)^{2^\star-1}dx=\int_{\mathcal{B}_0(\frac{1}{\hat\mu_\varepsilon})}\left(\left(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\right)^{n-2}\hat w_\varepsilon\left(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}x\right)\right)^{2^\star-1}dx$$

and thanks to (12.1.45) and (12.1.46), it follows that

$$\left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\right)^2 \int_{\mathcal{B}_0(\frac{1}{|y_\varepsilon|})} \hat{w}_\varepsilon(x)^{2^\star-1}dx \to \int_{I\!R^n} \left(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\right)^{\frac{n+2}{2}} dx$$

as $\varepsilon \to 0$. Coming back to (12.1.47), and since $|y_\varepsilon| \to 0$ as $\varepsilon \to 0$, we get that

$$B_\varepsilon \|u_\varepsilon\|_1 \left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\right)^{1-\frac{n}{2}} r_\varepsilon^{\frac{n}{2}+1} |y_\varepsilon|^{\frac{n}{2}+1} \to 0 \tag{12.1.48}$$

as $\varepsilon \to 0$. Noting that the sequence $(\hat{w}_\varepsilon)$ is bounded in any compact subset of $I\!R^n\backslash\{0\}$, it follows from standard elliptic theory, (12.1.44), and (12.1.48), that $\hat{w}_\varepsilon \to \Psi$ in $C^1_{loc}(I\!R^n\backslash\{0\})$, where Ψ is a solution of $\Delta\Psi = 0$ in $I\!R^n\backslash\{0\}$. We let $\delta > 0$ small, and we integrate (12.1.44) over $\mathcal{B}_0(\delta)$. Then,

$$\begin{aligned} &-\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \hat{w}_\varepsilon d\sigma - \alpha r_\varepsilon^2 |y_\varepsilon|^2 \int_{\mathcal{B}_0(\delta)} \hat{w}_\varepsilon dx \\ &+B_\varepsilon \|u_\varepsilon\|_1 \left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\right)^{1-\frac{n}{2}} r_\varepsilon^{\frac{n}{2}+1} |y_\varepsilon|^{\frac{n}{2}+1} |\mathcal{B}_0(\delta)| \\ &= \frac{1-\varepsilon}{K_n}\left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\right)^2 \int_{\mathcal{B}_0(\delta)} \hat{w}_\varepsilon^{2^\star-1}dx \end{aligned} \tag{12.1.47}$$

With the same arguments as above, it is easily seen that

$$r_\varepsilon^2 |y_\varepsilon|^2 \int_{\mathcal{B}_0(\delta)} \hat{w}_\varepsilon dx \to 0$$

and that

$$\left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\right)^2 \int_{\mathcal{B}_0(\delta)} \hat{w}_\varepsilon^{2^\star-1}dx \to \int_{I\!R^n} \left(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\right)^{\frac{n+2}{2}} dx$$

as $\varepsilon \to 0$. Since $\hat{w}_\varepsilon \to \Psi$ in $C^1_{loc}(I\!R^n)$, we also have that

$$\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \hat{w}_\varepsilon d\sigma \to \int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \Psi d\sigma$$

Passing to the limit as $\varepsilon \to 0$ in (12.1.49), it follows that

$$\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \Psi d\sigma + \frac{1}{K_n}\int_{I\!R^n} \left(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\right)^{\frac{n+2}{2}} dx = 0$$

and thus that

$$\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \Psi d\sigma + \frac{\omega_{n-1}}{n} A_n = 0 \tag{12.1.50}$$

where A_n is as in (12.1.36). In particular, $\Psi \not\equiv 0$. Independently, we have that $\Psi \geq 0$, and, thanks to (12.1.46), there exists $C > 0$ such that

$$|x|^{n-2}\Psi(x) \leq C \tag{12.1.51}$$

for all $x \in I\!R^n \backslash \{0\}$. Then the Kelvin transform $\tilde{\Psi}$ of Ψ given by

$$\tilde{\Psi}(x) = \frac{1}{|x|^{n-2}} \Psi\left(\frac{x}{|x|^2}\right)$$

is bounded and harmonic in $I\!R^n \backslash \{0\}$. In particular, $\tilde{\Psi}(x) = o(|x|^{2-n})$ as $|x| \to 0$. Thus, see for instance the excellent Han-Lin [23], 0 is a removable singularity for $\tilde{\Psi}$, and Liouville's theorem implies that $\tilde{\Psi}$ is constant. Hence, there exists $\lambda > 0$ such that $\Psi(x) = \lambda / |x|^{n-2}$ and, thanks to (12.1.50) we get that

$$\Psi(x) = \frac{A_n}{n(n-2)|x|^{n-2}}$$

where A_n is as in (12.1.36). In particular, taking $x = y_\varepsilon / |y_\varepsilon|$, we get that for any sequence (y_ε) in $\mathcal{B}$ such that $y_\varepsilon \to 0$ and $\frac{|y_\varepsilon|}{\hat{\mu}_\varepsilon} \to +\infty$ as $\varepsilon \to 0$,

$$|y_\varepsilon|^{n-2} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \hat{u}_\varepsilon(y_\varepsilon) \to \frac{A_n}{n(n-2)} = 2^{n-2} \omega_n^{\frac{2}{n}-1} \tag{12.1.52}$$

as $\varepsilon \to 0$. Combining (12.1.32), (12.1.37), (12.1.39), and (12.1.52), it follows that for any $\delta > 0$ and any $x \in \mathcal{B}_0(\delta)$,

$$\frac{1}{C(\delta)} \left(\frac{\hat{\mu}_\varepsilon}{\hat{\mu}_\varepsilon^2 + \frac{\omega_n^{2/n}}{4}|x|^2} \right)^{\frac{n-2}{2}} \le \hat{u}_\varepsilon(x) \le C(\delta) \left(\frac{\hat{\mu}_\varepsilon}{\hat{\mu}_\varepsilon^2 + \frac{\omega_n^{2/n}}{4}|x|^2} \right)^{\frac{n-2}{2}} \tag{12.1.53}$$

for $\varepsilon > 0$ small, where $C(\delta) > 1$ is such that $C(\delta) \to 1$ as $\delta \to 0$. When $n = 4$, and for $\delta > 0$ small, we get with (12.1.33) that

$$\int_{\mathcal{B} \backslash \mathcal{B}_0(\delta)} \hat{u}_\varepsilon^2 dx = O(\hat{\mu}_\varepsilon^2)$$

Thus,

$$\int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx = \int_{\mathcal{B}_0(\delta)} \hat{u}_\varepsilon^2 dx + O(\hat{\mu}_\varepsilon^2) \tag{12.1.54}$$

Independently,

$$\begin{aligned} \int_{\mathcal{B}_0(\delta)} \left(\frac{\hat{\mu}_\varepsilon}{\hat{\mu}_\varepsilon^2 + \frac{\omega_4^{1/2}}{4}|x|^2} \right)^2 dx &= \frac{16\omega_3}{\omega_4} \hat{\mu}_\varepsilon^2 \int_0^{\frac{\omega_4^{1/4}\delta}{2\hat{\mu}_\varepsilon}} \left(1 + r^2\right)^{-2} r^3 dr \\ &= \frac{16\omega_3}{\omega_4} \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o\left(\hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|\right) \end{aligned}$$

Then, coming back to (12.1.54), and thanks to (12.1.53), we get that

$$\int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx = \frac{16\omega_3}{\omega_4} \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o\left(\hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|\right) \tag{12.1.55}$$

Combining (12.1.40) and (12.1.55), this proves (12.1.43).

With independent arguments we also have that

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\|u_\varepsilon\|_1 = \hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}_0(r_\varepsilon)} u_\varepsilon dx = \hat{\mu}_\varepsilon^{1-\frac{n}{2}} r_\varepsilon^{\frac{n}{2}+1}\int_{\mathcal{B}} \hat{u}_\varepsilon dx$$

and (12.1.33), (12.1.37), and (12.1.39) imply that

$$r_\varepsilon^{-1-\frac{n}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\|u_\varepsilon\|_1 \to \int_{\mathcal{B}} \Phi dx = \frac{A_n\omega_{n-1}}{2n(n+2)}$$

as $\varepsilon \to 0$. By (12.1.35), we then get that

$$B_\varepsilon r_\varepsilon^{n+2} \to \frac{2n(n+2)}{\omega_{n-1}} \tag{12.1.56}$$

as $\varepsilon \to 0$. As already mentioned, we want to describe the behavior of B_ε in terms of ε as $\varepsilon \to 0$. Thanks to (12.1.42), (12.1.43), and (12.1.56), the question reduces to describing the behavior of $\hat{\mu}_\varepsilon$ in terms of ε as $\varepsilon \to 0$. This is the subject of the two following subsections.

12.1.5 Estimating $\hat{\mu}_\varepsilon$ in terms of ε (Part 1)

We want to describe the behavior of $\hat{\mu}_\varepsilon$ in terms of ε as $\varepsilon \to 0$. For that purpose, we let

$$\hat{u}_\varepsilon = (1+\theta_\varepsilon)U_\varepsilon + \hat{\mu}_\varepsilon^{\frac{n}{2}-1}(G_\varepsilon + w_\varepsilon) \tag{12.1.57}$$

where

$$\begin{aligned} U_\varepsilon &= \bar{\mu}_\varepsilon^{\frac{n}{2}-1}\left(\bar{\mu}_\varepsilon^2 + \frac{\omega_n^{2/n}}{4}(1-\varepsilon)|x|^2\right)^{1-\frac{n}{2}}, \\ G_\varepsilon &= \alpha_\varepsilon(|x|^2 - \beta_\varepsilon)\,, \\ \alpha_\varepsilon &= \frac{1}{2n}B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\,, \end{aligned} \tag{12.1.58}$$

θ_ε, β_ε are real numbers, $\bar{\mu}_\varepsilon$ is a positive real number and w_ε is a function. We choose θ_ε and $\bar{\mu}_\varepsilon$ such that

$$\begin{aligned} &\int_{\mathcal{B}} (\nabla U_\varepsilon, \nabla w_\varepsilon)\, dx = 0\,, \\ &\int_{\mathcal{B}} (\nabla\left(x, \nabla U_\varepsilon\right), \nabla w_\varepsilon)\, dx = 0 \end{aligned} \tag{12.1.59}$$

Let

$$U_\mu = \mu^{\frac{n}{2}-1}\left(\mu^2 + \frac{\omega_n^{2/n}}{4}(1-\varepsilon)|x|^2\right)^{1-\frac{n}{2}}$$

To get (12.1.59), it suffices to choose θ_ε and $\bar{\mu}_\varepsilon$ such that they minimize

$$J(\theta,\mu) = \int_{\mathcal{B}}\left|\nabla\left(\hat{u}_\varepsilon - (1+\theta)U_\mu\right) - 2\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\alpha_\varepsilon x\right|^2 dx$$

among the θ's in $\left[-\frac{1}{2},\frac{1}{2}\right]$ and the μ's in $\left[\frac{\hat{\mu}_\varepsilon}{2}, 2\hat{\mu}_\varepsilon\right]$ and to prove that θ_ε and $\bar{\mu}_\varepsilon$ lie in the interior of the interval of constraints for ε small enough. We prove indeed that

$$\theta_\varepsilon \to 0 \tag{12.1.60}$$

as $\varepsilon \to 0$ and that

$$\frac{\hat{\mu}_\varepsilon}{\bar{\mu}_\varepsilon} \to 1 \tag{12.1.61}$$

as $\varepsilon \to 0$. By (1.11), it is clear that $J(0, \hat{\mu}_\varepsilon) \to 0$ as $\varepsilon \to 0$ so that $J(\theta_\varepsilon, \bar{\mu}_\varepsilon) \to 0$ as $\varepsilon \to 0$. By (1.11) again, one gets that this enforces the following to happen:

$$\lim_{\varepsilon\to 0}\int_{\mathcal{B}} |\nabla\left((1+\theta_\varepsilon)U_\varepsilon + \hat{\mu}_\varepsilon^{\frac{n}{2}-1}G_\varepsilon\right)|^2 dx = \frac{1}{K_n^2}$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathcal{B}} \left(\nabla\left((1+\theta_\varepsilon)U_\varepsilon + \hat{\mu}_\varepsilon^{\frac{n}{2}-1}G_\varepsilon\right), \nabla\hat{u}_\varepsilon\right) dx = \frac{1}{K_n^2}$$

where U_ε and G_ε are as in (1.58). Using (1.11) once again, this is not difficult to check, noting that $\alpha_\varepsilon = O(1)$ thanks to (1.35), that these last two relations lead to (1.60) and (1.61). We also choose β_ε such that $w_\varepsilon = 0$ on $\partial\mathcal{B}$. Hence,

$$\alpha_\varepsilon(1-\beta_\varepsilon) + (1+\theta_\varepsilon)\left(\hat{\mu}_\varepsilon^2 + \frac{(1-\varepsilon)\omega_n^{2/n}}{4}\right)^{1-\frac{n}{2}} = 0$$

By (12.1.35) and (12.1.61), we have that

$$\begin{aligned} \alpha_\varepsilon &\to \frac{A_n}{2n} \\ \beta_\varepsilon &\to \frac{n}{n-2} \end{aligned} \tag{12.1.62}$$

as $\varepsilon \to 0$. Thanks to (12.1.33), (12.1.37), and (12.1.39),

$$\int_{\mathcal{B}} w_\varepsilon dx \to 0 \tag{12.1.63}$$

as $\varepsilon \to 0$. Let W_ε be such that

$$W_\varepsilon(x) = V_0\left((1-\varepsilon)^{1/2}x\right)$$

where V_0 is as above. Then,

$$\Delta W_\varepsilon = \frac{1-\varepsilon}{K_n} W_\varepsilon^{2^\star-1}$$

and

$$\begin{aligned} \int_{\mathcal{B}} |\nabla U_\varepsilon|^2 dx &= \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} |\nabla W_\varepsilon|^2 dx \\ \int_{I\!R^n} |\nabla W_\varepsilon|^2 dx &= (1-\varepsilon)^{1-\frac{n}{2}} K_n^{-1} \end{aligned}$$

As an easy consequence, writing that

$$\int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} |\nabla W_\varepsilon|^2 dx = \int_{I\!R^n} |\nabla W_\varepsilon|^2 dx - \int_{I\!R^n \setminus \mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} |\nabla W_\varepsilon|^2 dx$$

we get, thanks to (12.1.61), that

$$\int_{\mathcal{B}} |\nabla U_\varepsilon|^2 dx = \frac{1}{K_n} + \frac{n-2}{2K_n}\varepsilon - \frac{A_n^2 \omega_{n-1}}{n^2(n-2)} \hat{\mu}_\varepsilon^{n-2} + o(\hat{\mu}_\varepsilon^{n-2}) + o(\varepsilon) \tag{12.1.64}$$

Independent computations give that

$$\int_{\mathcal{B}} |\nabla G_\varepsilon|^2 dx = \frac{A_n^2 \omega_{n-1}}{n^2(n+2)} + o(1) \tag{12.1.65}$$

and

$$\int_{\mathcal{B}} (\nabla G_\varepsilon, \nabla w_\varepsilon)\, dx = \int_{\mathcal{B}} (\Delta G_\varepsilon) w_\varepsilon dx = o(1) \tag{12.1.66}$$

thanks to (12.1.63). Similarly, it is easily seen that

$$\int_{\mathcal{B}} (\nabla G_\varepsilon, \nabla U_\varepsilon)\, dx = -\frac{A_n^2 \omega_{n-1}}{2n^2} \hat{\mu}_\varepsilon^{\frac{n}{2}-1} + o(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}) \tag{12.1.67}$$

By (12.1.59), (12.1.61), and (12.1.64)-(12.1.67), we then get that

$$\begin{aligned} \int_{\mathcal{B}} |\nabla \hat{u}_\varepsilon|^2 dx &= \frac{1}{K_n} + \frac{2\theta_\varepsilon}{K_n} + \frac{(n-2)\varepsilon}{2K_n} - \frac{A_n^2 \omega_{n-1} \hat{\mu}_\varepsilon^{n-2}}{n^2-4} \\ &+ \hat{\mu}_\varepsilon^{n-2} \int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx + o(\theta_\varepsilon) + o(\varepsilon) + o(\hat{\mu}_\varepsilon^{n-2}) \end{aligned} \tag{12.1.68}$$

Now we claim that

$$\varepsilon = O(\hat{\mu}_\varepsilon^{n-2}) \tag{12.1.69}$$

Applying the sharp Sobolev inequality to $\hat{u}_\varepsilon$, we get thanks to (12.1.30) and (12.1.31) that

$$\frac{\varepsilon}{K_n} \le \alpha r_\varepsilon^2 \int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx - B_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \int_{\mathcal{B}} \hat{u}_\varepsilon dx \tag{12.1.70}$$

By (12.1.40),

$$\alpha r_\varepsilon^2 \int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx = \frac{\omega_{n-1}}{4n} A_n^2 \hat{\mu}_\varepsilon^{n-2} + o(\hat{\mu}_\varepsilon^{n-2}) \tag{12.1.71}$$

while (12.1.33), (12.1.35), (12.1.37), and (12.1.39) imply that

$$B_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \int_{\mathcal{B}} \hat{u}_\varepsilon dx = \frac{\omega_{n-1}}{2n(n+2)} A_n^2 \hat{\mu}_\varepsilon^{n-2} + o(\hat{\mu}_\varepsilon^{n-2}) \tag{12.1.72}$$

Combining (12.1.70)-(12.1.72), this proves (12.1.69).

Let us now multiply (12.1.30) by $\hat{u}_\varepsilon$ and integrate over $\mathcal{B}$. Thanks to (12.1.68)-(12.1.72), we get that

$$\frac{n\varepsilon}{2K_n} + \frac{2\theta_\varepsilon}{K_n} + \hat{\mu}_\varepsilon^{n-2} \int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx + o(\theta_\varepsilon) = \frac{\omega_{n-1} A_n^2}{4(n-2)} \hat{\mu}_\varepsilon^{n-2} + o(\hat{\mu}_\varepsilon^{n-2}) \tag{12.1.73}$$

In particular,

$$\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx = o\left(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}\right) + O(1)$$

and, thanks to the Sobolev inequality,

$$\left(\int_{\mathcal{B}} |w_\varepsilon|^{2^\star} dx\right)^{\frac{2}{2^\star}} = o\left(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}\right) + O(1) \tag{12.1.74}$$

For $1 \le p \le 3$ and X, Y such that $X \ge 0$ and $X + Y \ge 0$,

$$(X+Y)^p = X^p + pX^{p-1}Y + \frac{p(p-1)}{2}X^{p-2}Y^2 + O(|Y|^p)$$

while for $3 \le p \le 4$ and X, Y as above,

$$(X+Y)^p = X^p + pX^{p-1}Y + \frac{p(p-1)}{2}X^{p-2}Y^2 + O(X^{p-3}|Y|^3) + O(|Y|^p)$$

Writing that $\int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star} dx = 1$, we then get that

$$\begin{aligned} 1 &= (1+\theta_\varepsilon)^{2^\star} \int_{\mathcal{B}} U_\varepsilon^{2^\star} dx + 2^\star (1+\theta_\varepsilon)^{2^\star-1} \hat{\mu}_\varepsilon^{\frac{n}{2}-1} \int_{\mathcal{B}} U_\varepsilon^{2^\star-1}(G_\varepsilon + w_\varepsilon) dx \\ &+ \frac{2^\star(2^\star-1)}{2} \hat{\mu}_\varepsilon^{n-2} (1+\theta_\varepsilon)^{2^\star-2} \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}(G_\varepsilon + w_\varepsilon)^2 dx \\ &+ O\left(\hat{\mu}_\varepsilon^n \int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star} dx\right) \end{aligned} \tag{12.1.75}$$

if $n \ge 6$, and

$$\begin{aligned} 1 &= (1+\theta_\varepsilon)^{2^\star} \int_{\mathcal{B}} U_\varepsilon^{2^\star} dx + 2^\star (1+\theta_\varepsilon)^{2^\star-1} \hat{\mu}_\varepsilon^{\frac{n}{2}-1} \int_{\mathcal{B}} U_\varepsilon^{2^\star-1}(G_\varepsilon + w_\varepsilon) dx \\ &+ \frac{2^\star(2^\star-1)}{2} \hat{\mu}_\varepsilon^{n-2} (1+\theta_\varepsilon)^{2^\star-2} \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}(G_\varepsilon + w_\varepsilon)^2 dx \\ &+ O\left(\hat{\mu}_\varepsilon^{\frac{3n}{2}-3} \left(\int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star} dx\right)^{3/2^\star}\right) + O\left(\hat{\mu}_\varepsilon^n \int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star} dx\right) \end{aligned} \tag{12.1.76}$$

if $n = 4, 5$. For W_ε as above,

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star} dx = \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} W_\varepsilon^{2^\star} dx \quad \text{and} \quad \int_{\mathbb{R}^n} W_\varepsilon^{2^\star} dx = \frac{1}{(1-\varepsilon)^{n/2}}$$

Thanks to (12.1.61) and (12.1.69) we then get that

$$1 - (1+\theta_\varepsilon)^{2^\star} \int_{\mathcal{B}} U_\varepsilon^{2^\star} dx = -2^\star \theta_\varepsilon - \frac{n}{2}\varepsilon + o(\theta_\varepsilon) + o(\hat{\mu}_\varepsilon^{n-2}) \tag{12.1.77}$$

By (12.1.74) we easily get that

$$\hat{\mu}_\varepsilon^n \int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star} dx = o(\hat{\mu}_\varepsilon^{n-2}) + o(\theta_\varepsilon) \tag{12.1.78}$$

and

$$\hat{\mu}_\varepsilon^{\frac{3n}{2}-3} \left(\int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star} dx\right)^{3/2^\star} = o(\hat{\mu}_\varepsilon^{n-2}) + o(\theta_\varepsilon) \tag{12.1.79}$$

Independently, it is easily checked that $\hat{\mu}_\varepsilon^{1-\frac{n}{2}} W_\varepsilon(\frac{1}{\hat{\mu}_\varepsilon}x) = U_\varepsilon(x)$. Hence

$$\Delta U_\varepsilon = \frac{1-\varepsilon}{K_n} U_\varepsilon^{2^\star-1}$$

and, thanks to (12.1.59), we get that

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-1}(G_\varepsilon + w_\varepsilon)dx = \int_{\mathcal{B}} U_\varepsilon^{2^\star-1} G_\varepsilon dx$$

Then,

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-1}(G_\varepsilon + w_\varepsilon)dx = \hat{\mu}_\varepsilon^{\frac{n}{2}-1} \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} W_\varepsilon(x)^{2^\star-1} G_\varepsilon(\hat{\mu}_\varepsilon x)dx$$

and we find, thanks to (12.1.61), that

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-1}(G_\varepsilon + w_\varepsilon)dx = -\frac{A_n^2 K_n \omega_{n-1}}{2n(n-2)} \hat{\mu}_\varepsilon^{\frac{n}{2}-1} + o(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}) \tag{12.1.80}$$

Independently, it is easily seen with (12.1.74) that

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}(G_\varepsilon + w_\varepsilon)^2 dx = \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dx + o(1) + o(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}) \tag{12.1.81}$$

Coming back to (12.1.75) and (12.1.76), we get with (12.1.77)-(12.1.81) that

$$\begin{aligned} \hat{\mu}_\varepsilon^{n-2} \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dx &= -\frac{2(n-2)}{n+2}\theta_\varepsilon - \frac{(n-2)^2}{2(n+2)}\varepsilon \\ &+ \frac{A_n^2 K_n \omega_{n-1}}{n(n+2)} \hat{\mu}_\varepsilon^{n-2} + o(\theta_\varepsilon) + o(\hat{\mu}_\varepsilon^{n-2}) \end{aligned} \tag{12.1.82}$$

On such an assertion, note that

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dx = O\left(\left(\int_{\mathcal{B}} w_\varepsilon^{2^\star} dx\right)^{2/2^\star}\right)$$

Independently, it is easily seen from (12.1.30) that

$$\Delta w_\varepsilon = \frac{1-\varepsilon}{K_n}\left(\hat{u}_\varepsilon^{2^\star-1} - (1+\theta_\varepsilon)U_\varepsilon^{2^\star-1}\right)\hat{\mu}_\varepsilon^{1-\frac{n}{2}} + \alpha r_\varepsilon^2 \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \hat{u}_\varepsilon$$

in $\mathcal{B}$. By (12.1.59) we then get that

$$\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx = \frac{1-\varepsilon}{K_n}\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star-1} w_\varepsilon dx + \alpha r_\varepsilon^2 \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{\mathcal{B}} \hat{u}_\varepsilon w_\varepsilon dx \tag{12.1.83}$$

Now we want to estimate the terms in the right hand side of (12.1.83). By (12.1.37) and (12.1.39), $|x|^{n-2}w_\varepsilon(x) \to 0$ in $C^0_{loc}(\overline{\mathcal{B}}\backslash\{0\})$ as $\varepsilon \to 0$. Independently, it follows from (12.1.53) that for $|x| \le \delta$ and ε small, $|x|^{n-2}w_\varepsilon(x) \le \varepsilon(\delta)$ where $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Hence,

$$|x|^{n-2}w_\varepsilon(x) \to 0 \text{ in } C^0(\overline{\mathcal{B}}) \tag{12.1.84}$$

as $\varepsilon \to 0$. We now write that

$$\int_{\mathcal{B}} \frac{\hat{u}_\varepsilon}{|x|^{n-2}} dx = \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} \frac{\hat{u}_\varepsilon}{|x|^{n-2}} dx + \int_{\mathcal{B}\backslash\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} \frac{\hat{u}_\varepsilon}{|x|^{n-2}} dx$$

Thanks to (12.1.42) and (12.1.43), it follows from (12.1.32) that

$$r_\varepsilon^2 \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} \frac{\hat{u}_\varepsilon}{|x|^{n-2}} dx = O(1)$$

and from (12.1.33) that

$$r_\varepsilon^2 \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{\mathcal{B}\backslash\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} \frac{\hat{u}_\varepsilon}{|x|^{n-2}} dx = O(1)$$

Then, (12.1.84) implies that

$$r_\varepsilon^2 \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{\mathcal{B}} \hat{u}_\varepsilon w_\varepsilon dx = o(1) \tag{12.1.85}$$

For X, Y such that $X \geq 0$ and $X + Y \geq 0$ we write now that

$$(X+Y)^{2^\star-1} = X^{2^\star-1} + (2^\star - 1)X^{2^\star-2}Y + f(n)O(X^{2^\star-3}Y^2) + O(|Y|^{2^\star-1})$$

where $f(n) = 1$ if $n = 4, 5$, and $f(n) = 0$ if $n \geq 6$. Then,

$$\begin{aligned}
\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star-1} w_\varepsilon dx &= (1+\theta_\varepsilon)^{2^\star-1} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{\mathcal{B}} U_\varepsilon^{2^\star-1} w_\varepsilon dx \\
&+ (2^\star - 1)(1+\theta_\varepsilon)^{2^\star-2} \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}(G_\varepsilon + w_\varepsilon) w_\varepsilon dx \\
&+ \hat{\mu}_\varepsilon^{\frac{n}{2}-1} O\left(\int_{\mathcal{B}} U_\varepsilon^{2^\star-3}(G_\varepsilon + w_\varepsilon)^2 |w_\varepsilon| dx\right) f(n) \\
&+ \hat{\mu}_\varepsilon^2 O\left(\int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star-1} |w_\varepsilon| dx\right)
\end{aligned} \tag{12.1.86}$$

As when proving (12.1.81), it follows from (12.1.74) that

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}(G_\varepsilon + w_\varepsilon) w_\varepsilon dx = \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dx + o(1) + o(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}) \tag{12.1.87}$$

Still thanks to (12.1.74), we easily get that

$$\int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star-1} |w_\varepsilon| dx = O(1) + O(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}) \tag{12.1.88}$$

and

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-3}(G_\varepsilon + w_\varepsilon)^2 |w_\varepsilon| dx = O(1) + O(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}) \tag{12.1.89}$$

when $n = 4, 5$. We have already seen that

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-1} w_\varepsilon dx = 0 \tag{12.1.90}$$

Combining (12.1.86)-(12.1.90), it follows that

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star-1} w_\varepsilon dx = \frac{n+2}{n-2} \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dx + o(1) + o(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}) \tag{12.1.91}$$

Coming back to (12.1.83), we get with (12.1.85) and (12.1.91) that

$$\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx = \frac{n+2}{(n-2)K_n} \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dx + o(1) + o(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}) \tag{12.1.92}$$

Then, combining (12.1.82) with (12.1.92),

$$\begin{aligned} \hat{\mu}_\varepsilon^{n-2} \int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx = -\frac{2}{K_n}\theta_\varepsilon - \frac{n-2}{2K_n}\varepsilon \\ + \frac{A_n^2 \omega_{n-1}}{n(n-2)} \hat{\mu}_\varepsilon^{n-2} + o(\theta_\varepsilon) + o(\hat{\mu}_\varepsilon^{n-2}) \end{aligned} \tag{12.1.93}$$

and coming back to (12.1.73), we get that

$$\varepsilon = \frac{(n-4)\omega_{n-1}}{4n(n-2)} A_n^2 K_n \hat{\mu}_\varepsilon^{n-2} + o(\theta_\varepsilon) + o(\hat{\mu}_\varepsilon^{n-2}) \tag{12.1.94}$$

where A_n is given by (12.1.36).

As already mentioned, we want to express $\hat{\mu}_\varepsilon$ in terms of ε as $\varepsilon \to 0$. Thanks to (12.1.94), if we prove that $\theta_\varepsilon = O(\hat{\mu}_\varepsilon^{n-2})$, then we get a description of $\hat{\mu}_\varepsilon$ in terms of ε as $\varepsilon \to 0$. The following section is devoted to this estimation of θ_ε in terms of $\hat{\mu}_\varepsilon$.

12.1.6 Estimating $\hat{\mu}_\varepsilon$ in terms of ε (Part 2)

After (12.1.94), we claim that

$$\theta_\varepsilon = O(\hat{\mu}_\varepsilon^{n-2}) \tag{12.1.95}$$

We prove (12.1.95) by contradiction. We assume that

$$|\theta_\varepsilon| \hat{\mu}_\varepsilon^{2-n} \to +\infty \tag{12.1.96}$$

as $\varepsilon \to 0$. Then, by (12.1.69), (12.1.73) and (12.1.82),

$$\lim_{\varepsilon \to 0} \frac{\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx}{\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dx} = \frac{2^\star - 1}{K_n} \tag{12.1.97}$$

We contradict (12.1.97). For that purpose we consider the eigenvalue problem

$$\begin{aligned} &\Delta \varphi_{i,\varepsilon} = \mu_{i,\varepsilon} U_\varepsilon^{2^\star-2} \varphi_{i,\varepsilon} \text{ in } \mathcal{B} \\ &\varphi_{i,\varepsilon} = 0 \text{ on } \partial\mathcal{B} \end{aligned} \tag{12.1.98}$$

where

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} \varphi_{i,\varepsilon} \varphi_{j,\varepsilon} dx = \delta_{ij}$$

and $\mu_{1,\varepsilon} \leq \ldots \leq \mu_{i,\varepsilon} \leq \ldots$. Let V_0 be as above, given by

$$V_0(x) = \left(1 + \frac{\omega_n^{2/n}}{4}|x|^2\right)^{1-\frac{n}{2}}$$

We claim that for any $i \geq 1$,

$$\mu_{i,\varepsilon} \to \mu_i \tag{12.1.99}$$

as $\varepsilon \to 0$, $\mu_1 \leq \ldots \leq \mu_i \leq \ldots$, and that

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star - 2} \left(\varphi_{i,\varepsilon} - \psi_{i,\varepsilon}\right)^2 dx \to 0 \tag{12.1.100}$$

as $\varepsilon \to 0$ for functions $\psi_{i,\varepsilon}$ satisfying that

$$\hat{\mu}_\varepsilon^{\frac{n}{2}-1} \psi_{i,\varepsilon}(\hat{\mu}_\varepsilon x) \to \psi_i(x)$$

in $C^0_{loc}(I\!R^n) \cap L^{2^\star}(I\!R^n)$ as $\varepsilon \to 0$, where the ψ_i's are such that

$$\begin{gathered} \Delta \psi_i = \mu_i V_0^{2^\star - 2} \psi_i \ \text{ in } I\!R^n \\ \int_{I\!R^n} V_0^{2^\star - 2} \psi_i^2 dx < +\infty \end{gathered} \tag{12.1.101}$$

We prove (12.1.99) and (12.1.100) by induction. When $i = 1$,

$$\mu_{1,\varepsilon} = \inf_{\{\varphi \in C_c^\infty(\mathcal{B}), \varphi \not\equiv 0\}} \frac{\int_{\mathcal{B}} |\nabla \varphi|^2 dx}{\int_{\mathcal{B}} U_\varepsilon^{2^\star - 2} \varphi^2 dx}$$

On the one hand, taking $\varphi = U_\varepsilon - U_\varepsilon(1)$, we get that

$$\limsup_{\varepsilon \to 0} \mu_{1,\varepsilon} \leq \frac{1}{K_n}$$

On the other hand, thanks to the sharp Sobolev inequality,

$$\begin{aligned} \left(\int_{\mathcal{B}} |\varphi_{1,\varepsilon}|^{2^\star} dx\right)^{\frac{2}{2^\star}} &\leq K_n \int_{\mathcal{B}} |\nabla \varphi_{1,\varepsilon}|^2 dx \\ &= K_n \mu_{1,\varepsilon} \int_{\mathcal{B}} U_\varepsilon^{2^\star - 2} \varphi_{1,\varepsilon}^2 dx \\ &\leq K_n \mu_{1,\varepsilon} \left(\int_{\mathcal{B}} U_\varepsilon^{2^\star} dx\right)^{1 - \frac{2}{2^\star}} \left(\int_{\mathcal{B}} |\varphi_{1,\varepsilon}|^{2^\star} dx\right)^{\frac{2}{2^\star}} \end{aligned}$$

and we get that

$$\liminf_{\varepsilon \to 0} \mu_{1,\varepsilon} \geq \frac{1}{K_n}$$

Hence,

$$\mu_{1,\varepsilon} \to \mu_1 = \frac{1}{K_n} \tag{12.1.102}$$

as $\varepsilon \to 0$, and we also have that

$$\int_{\mathcal{B}} |\varphi_{1,\varepsilon}|^{2^\star} dx \to 1 \ \text{ and } \ \int_{\mathcal{B}} |\nabla \varphi_{1,\varepsilon}|^2 dx \to \frac{1}{K_n}$$

as $\varepsilon \to 0$, since $\int_{\mathcal{B}} U_\varepsilon^{2^\star - 2} \varphi_{1,\varepsilon}^2 dx = 1$. We let W_ε be as above, given by

$$W_\varepsilon(x) = \left(1 + \frac{\omega_n^{2/n}}{4}(1 - \varepsilon)|x|^2\right)^{1 - \frac{n}{2}}$$

and let $\hat{\varphi}_{1,\varepsilon}$ be given by

$$\hat{\varphi}_{1,\varepsilon}(x) = \hat{\mu}_\varepsilon^{\frac{n}{2}-1}\varphi_{1,\varepsilon}(\hat{\mu}_\varepsilon x) \quad \text{in } \mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})$$
$$\hat{\varphi}_{1,\varepsilon}(x) = 0 \quad \text{in } I\!R^n \backslash \mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})$$

It is easily seen that

$$\int_{I\!R^n} W_\varepsilon^{2^\star-2}\hat{\varphi}_{1,\varepsilon}^2 dx = \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\varphi_{1,\varepsilon}^2 dx = 1$$

and that the $\hat{\varphi}_{1,\varepsilon}$'s are bounded in $D_1^2(I\!R^n)$. We may therefore assume that the $\hat{\varphi}_{1,\varepsilon}$'s converge weakly to some ψ_1 in $D_1^2(I\!R^n)$ as $\varepsilon \to 0$. In particular, it is easily seen that

$$\int_{I\!R^n} \psi_1^{2^\star} dx \le 1 \tag{12.1.103}$$

For $R > 0$, we write that

$$\begin{aligned} 1 = \int_{I\!R^n} W_\varepsilon^{2^\star-2}\hat{\varphi}_{1,\varepsilon}^2 dx &= \int_{I\!R^n} (W_\varepsilon^{2^\star-2} - V_0^{2^\star-2})\hat{\varphi}_{1,\varepsilon}^2 dx \\ &\quad + \int_{\mathcal{B}_0(R)} V_0^{2^\star-2}\hat{\varphi}_{1,\varepsilon}^2 dx + \int_{I\!R^n\backslash\mathcal{B}_0(R)} V_0^{2^\star-2}\hat{\varphi}_{1,\varepsilon}^2 dx \end{aligned}$$

By Hölder's inequality,

$$\int_{I\!R^n} (W_\varepsilon^{2^\star-2} - V_0^{2^\star-2})\hat{\varphi}_{1,\varepsilon}^2 dx \le C\left(\int_{I\!R^n} \left|W_\varepsilon^{2^\star-2} - V_0^{2^\star-2}\right|^{2^\star/(2^\star-2)} dx\right)^{1-\frac{2}{2^\star}}$$

and

$$\int_{I\!R^n\backslash\mathcal{B}_0(R)} V_0^{2^\star-2}\hat{\varphi}_{1,\varepsilon}^2 dx \le C\left(\int_{I\!R^n\backslash\mathcal{B}_0(R)} V_0^{2^\star} dx\right)^{(2^\star-2)/2^\star}$$

Hence,

$$\limsup_{\varepsilon\to 0} \int_{I\!R^n} (W_\varepsilon^{2^\star-2} - V_0^{2^\star-2})\hat{\varphi}_{1,\varepsilon}^2 dx = 0$$
$$\lim_{R\to+\infty} \limsup_{\varepsilon\to 0} \int_{I\!R^n\backslash\mathcal{B}_0(R)} V_0^{2^\star-2}\hat{\varphi}_{1,\varepsilon}^2 dx = 0$$

and we get that

$$\int_{I\!R^n} V_0^{2^\star-2}\psi_1^2 dx = \lim_{R\to+\infty} \int_{\mathcal{B}_0(R)} V_0^{2^\star-2}\psi_1^2 dx = 1$$

By Hölder's inequality,

$$\int_{I\!R^n} V_0^{2^\star-2}\psi_1^2 dx \le \left(\int_{I\!R^n} V_0^{2^\star} dx\right)^{(2^\star-2)/2^\star} \left(\int_{I\!R^n} \psi_1^{2^\star} dx\right)^{2/2^\star} \tag{12.1.104}$$

and since V_0 is of norm 1 in $L^{2^\star}(I\!R^n)$, we get from (12.1.103) and (12.1.104) that ψ_1 is of norm 1 in $L^{2^\star}(I\!R^n)$ and that $\psi_1 = V_0$. Then ψ_1 is a solution of (12.1.101). Writing that

$$\int_{I\!R^n} |\nabla(\hat{\varphi}_{1,\varepsilon} - \psi_1)|^2 dx = \int_{I\!R^n} |\nabla\hat{\varphi}_{1,\varepsilon}|^2 dx + \int_{I\!R^n} |\nabla\psi_1|^2 dx - \frac{2}{K_n}\int_{I\!R^n} \hat{\varphi}_{1,\varepsilon}\psi_1^{2^\star-1} dx$$

we also get that the $\hat{\varphi}_{1,\varepsilon}$'s converge strongly to ψ_1 in $D_1^2(I\!R^n)$, and in particular that the $\hat{\varphi}_{1,\varepsilon}$'s converge strongly to ψ_1 in $L^{2^\star}(I\!R^n)$. Then, (12.1.100) is proved and we get the result for $i = 1$. Let us now assume that (12.1.99)-(12.1.101) hold for $i = 1, \dots, p$. We have

$$\mu_{p+1,\varepsilon} = \inf_{\varphi \in \mathcal{H}} \int_{\mathcal{B}} |\nabla\varphi|^2 dx$$

where $\mathcal{H}$ is the set of the functions $\varphi \in C_c^\infty(\mathcal{B})$ which are such that

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\varphi^2 dx = 1 \text{ and } \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\varphi_{i,\varepsilon}\varphi dx = 0$$

for all $i = 1, \dots, p$. We claim first that the $\mu_{p+1,\varepsilon}$'s are bounded. It is easily seen that the $\hat{\varphi}_{i,\varepsilon}$'s, $i = 1, \dots, p$, are bounded in $D_1^2(I\!R^n)$. Then, it follows from (12.1.100) that the $\hat{\varphi}_{i,\varepsilon}$'s, $i = 1, \dots, p$, converge to ψ_i weakly in $D_1^2(I\!R^n)$. We let $f \in C_c^\infty(I\!R^n)$ be such that

$$\int_{I\!R^n} V_0^{2^\star-2} f\psi_i dx = 0$$

for all $i = 1, \dots, p$. We set

$$f_\varepsilon(x) = \hat{\mu}_\varepsilon^{1-\frac{n}{2}} f\left(\frac{1}{\hat{\mu}_\varepsilon}x\right)$$

and

$$\tilde{f}_\varepsilon = f_\varepsilon - \sum_{i=1}^{p}\left(\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} f_\varepsilon\varphi_{i,\varepsilon}dx\right)\varphi_{i,\varepsilon}$$

For $\varepsilon > 0$ sufficiently small, $\tilde{f}_\varepsilon \in C_c^\infty(\mathcal{B})$, and since $\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\varphi_{i,\varepsilon}\varphi_{j,\varepsilon}dx = \delta_{ij}$, we have that

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\tilde{f}_\varepsilon\varphi_{i,\varepsilon}dx = 0 \tag{12.1.105}$$

for all $i = 1, \dots, p$. It is easily checked that

$$\begin{aligned} \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\tilde{f}_\varepsilon^2 dx &= \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} f_\varepsilon^2 dx - \sum_{i=1}^{p}\left(\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} f_\varepsilon\varphi_{i,\varepsilon}dx\right)^2 \\ \int_{\mathcal{B}} |\nabla\tilde{f}_\varepsilon|^2 dx &= \int_{\mathcal{B}} |\nabla f_\varepsilon|^2 dx - \sum_{i=1}^{p}\left(\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} f_\varepsilon\varphi_{i,\varepsilon}dx\right)^2 \mu_{i,\varepsilon} \end{aligned} \tag{12.1.106}$$

for all $\varepsilon > 0$, and that

$$\begin{aligned} \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} f_\varepsilon^2 dx &= \int_{I\!R^n} W_\varepsilon^{2^\star-2} f^2 dx \to \int_{I\!R^n} V_0^{2^\star-2} f^2 dx \\ \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} f_\varepsilon\varphi_{i,\varepsilon}dx &= \int_{I\!R^n} W_\varepsilon^{2^\star-2} f\hat{\varphi}_{i,\varepsilon}dx \to \int_{I\!R^n} V_0^{2^\star-2} f\psi_i dx = 0 \end{aligned} \tag{12.1.107}$$

as $\varepsilon \to 0$. Since,

$$\int_{\mathcal{B}} |\nabla f_\varepsilon|^2 dx = \int_{I\!R^n} |\nabla f|^2 dx \tag{12.1.108}$$

for all $\varepsilon > 0$, we get by combining (12.1.105)-(12.1.108) that for $C > 1$ and $\varepsilon > 0$ small,

$$\mu_{p+1,\varepsilon} \le C\frac{\int_{I\!R^n} |\nabla f|^2 dx}{\int_{I\!R^n} V_0^{2^\star-2} f^2 dx}$$

In particular, the $\mu_{p+1,\varepsilon}$'s are bounded, and this proves the above claim. We may then assume that $\mu_{p+1,\varepsilon} \to \mu_{p+1}$ as $\varepsilon \to 0$, where $\mu_{p+1} \geq \mu_p$. As above, the $\hat{\varphi}_{p+1,\varepsilon}$'s are bounded in $D_1^2(I\!R^n)$. We may therefore assume that the $\hat{\varphi}_{p+1,\varepsilon}$'s converge weakly to some ψ_{p+1} in $D_1^2(I\!R^n)$. The $\hat{\varphi}_{p+1,\varepsilon}$'s are solutions of

$$\Delta \hat{\varphi}_{p+1,\varepsilon} = \mu_{p+1,\varepsilon} W_\varepsilon^{2^\star-2} \hat{\varphi}_{p+1,\varepsilon}$$

in $\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})$. It is then clear that ψ_{p+1} is a solution of (12.1.101). Now we write that

$$\int_{I\!R^n} W_\varepsilon^{2^\star-2}(\hat{\varphi}_{p+1,\varepsilon} - \psi_{p+1})^2 dx = \int_{\mathcal{B}_0(R)} W_\varepsilon^{2^\star-2}(\hat{\varphi}_{p+1,\varepsilon} - \psi_{p+1})^2 dx + o_{R,\varepsilon}(1)$$

where

$$\lim_{R\to+\infty} \lim_{\varepsilon\to 0} o_{R,\varepsilon}(1) = 0$$

We may assume that the $\hat{\varphi}_{p+1,\varepsilon}$'s converge to ψ_{p+1} in $L^2_{loc}(I\!R^n)$. Hence

$$\int_{I\!R^n} W_\varepsilon^{2^\star-2}(\hat{\varphi}_{p+1,\varepsilon} - \psi_{p+1})^2 dx \to 0$$

as $\varepsilon \to 0$, and this clearly proves that (12.1.100) holds. By induction, it follows that (12.1.99)-(12.1.101) hold for all i. Now, as shown by Bianchi-Egnell [4] and Rey [34], the eigenvalue problem

$$\begin{aligned} &\Delta\psi = \nu V_0^{2^\star-2}\psi \text{ in } I\!R^n \\ &\int_{I\!R^n} V_0^{2^\star-2}\psi^2 dx < +\infty \end{aligned} \tag{12.1.109}$$

has a discrete spectrum $\nu_1 \leq \ldots \leq \nu_i \leq \ldots$ such that

$$\nu_1 = \frac{1}{K_n} \;\; , \;\; \nu_2 = \ldots = \nu_{n+2} = \frac{2^\star - 1}{K_n} \;\; , \;\; \nu_{n+3} > \frac{2^\star-1}{K_n}$$

and the eigenspaces corresponding to the eigenvalues $\frac{1}{K_n}$ and $\frac{2^\star-1}{K_n}$ are

$$\mathcal{E}_1 = \mathrm{Span}\{V_0\} \;\text{ and }\; \mathcal{E}_2 = \mathrm{Span}\{\Phi_j, j = 0,\ldots,n\}$$

where

$$\Phi_0 = \left(1 + \frac{\omega_n^{2/n}}{4}|x|^2\right)^{-\frac{n}{2}} \left(1 - \frac{\omega_n^{2/n}}{4}|x|^2\right) \text{ and } \Phi_j = \left(1 + \frac{\omega_n^{2/n}}{4}|x|^2\right)^{-\frac{n}{2}} x_j$$

for $j = 1,\ldots,n$. Coming back to our problem, we let k_0 be such that $\mu_{k_0+1} > \frac{2^\star-1}{K_n}$, and write that

$$w_\varepsilon = \sum_{i=1}^{k_0} \alpha_{i,\varepsilon}\varphi_{i,\varepsilon} + R_\varepsilon \tag{12.1.110}$$

where w_ε is given by (12.1.58), and

$$\alpha_{i,\varepsilon} = \frac{\int_{\mathcal{B}} (\nabla w_\varepsilon, \nabla\varphi_{i,\varepsilon})\, dx}{\int_{\mathcal{B}} |\nabla\varphi_{i,\varepsilon}|^2 dx} = \frac{1}{\mu_{i,\varepsilon}} \int_{\mathcal{B}} (\nabla w_\varepsilon, \nabla\varphi_{i,\varepsilon})\, dx$$

We write that

$$\left(\int_{\mathcal{B}} (\nabla w_\varepsilon, \nabla \varphi_{i,\varepsilon})\, dx\right)^2 \le 2\left(\int_{\mathcal{B}} (\nabla w_\varepsilon, \nabla \psi_{i,\varepsilon})\, dx\right)^2$$
$$+2\left(\int_{\mathcal{B}} (\nabla w_\varepsilon, \nabla(\varphi_{i,\varepsilon} - \psi_{i,\varepsilon}))\, dx\right)^2$$

where $\psi_{1,\varepsilon} = U_\varepsilon$ and $\psi_{i,\varepsilon}(x) = \bar{\mu}_\varepsilon^{1-\frac{n}{2}} \psi_i(\frac{\sqrt{1-\varepsilon}}{\bar{\mu}_\varepsilon} x)$ for $2 \le i \le k_0$, so that the $\psi_{i,\varepsilon}$'s when $2 \le i \le k_0$ are linear combinations of the functions $\hat{\Phi}_j$ given by

$$\hat{\Phi}_j(x) = \bar{\mu}_\varepsilon^{1-\frac{n}{2}} \Phi_j(\frac{\sqrt{1-\varepsilon}}{\bar{\mu}_\varepsilon} x)$$

By (12.1.59), and since the w_ε's are radially symmetrical,

$$\int_{\mathcal{B}} (\nabla w_\varepsilon, \nabla \psi_{i,\varepsilon})\, dx = 0$$

for $1 \le i \le k_0$. Hence,

$$\alpha_{i,\varepsilon}^2 \le \frac{2}{\mu_{i,\varepsilon}^2} \int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx \int_{\mathcal{B}} |\nabla(\varphi_{i,\varepsilon} - \psi_{i,\varepsilon})|^2 dx \tag{12.1.111}$$

When $i = 1$, we have seen that the $\hat{\varphi}_{1,\varepsilon}$'s converge strongly to V_0 in $D_1^2(I\!R^n)$ as $\varepsilon \to 0$. Hence,

$$\int_{\mathcal{B}} |\nabla(\varphi_{1,\varepsilon} - \psi_{1,\varepsilon})|^2 dx \to 0 \tag{12.1.112}$$

as $\varepsilon \to 0$. We claim now that for $2 \le i \le k_0$,

$$\int_{\mathcal{B}} |\nabla(\varphi_{i,\varepsilon} - \psi_{i,\varepsilon})|^2 dx \to 0 \tag{12.1.113}$$

It is easily seen that

$$\begin{aligned}\int_{\mathcal{B}} |\nabla(\varphi_{i,\varepsilon} - \psi_{i,\varepsilon})|^2 dx &= \int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})} |\nabla(\hat{\varphi}_{i,\varepsilon} - \psi_i)|^2 dx \\ &= \mu_{i,\varepsilon} + \int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})} |\nabla \psi_i|^2 dx - 2\int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})} (\nabla \hat{\varphi}_{i,\varepsilon}, \nabla \psi_i)\, dx\end{aligned} \tag{12.1.114}$$

Similarly,

$$\int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})} W_\varepsilon^{2^\star-2} \psi_i^2 dx = \int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})} V_0^{2^\star-2} \psi_i^2 dx + o(1) \tag{12.1.115}$$

and

$$\int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})} W_\varepsilon^{2^\star-2} \hat{\varphi}_{i,\varepsilon} \psi_i dx = \int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})} V_0^{2^\star-2} \hat{\varphi}_{i,\varepsilon} \psi_i dx + o(1) \tag{12.1.116}$$

Independently, since the ψ_i's are linear combinations of the Φ_j's, we get that

$$\int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})} |\nabla \psi_i|^2 dx = \mu_i \int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})} V_0^{2^\star-2} \psi_i^2 dx + o(1) \tag{12.1.117}$$

At last, it follows from (12.1.100) that

$$1+\int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} W_\varepsilon^{2^\star-2}\psi_i^2 dx = 2\int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} W_\varepsilon^{2^\star-2}\hat{\varphi}_{i,\varepsilon}\psi_i dx + o(1) \tag{12.1.118}$$

Noting that

$$\int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} (\nabla\hat{\varphi}_{i,\varepsilon}, \nabla\psi_i)\, dx = \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} \hat{\varphi}_{i,\varepsilon}\Delta\psi_i dx \tag{12.1.119}$$

we get by combining (12.1.114)-(12.1.119) that (12.1.113) holds. Coming back to (12.1.111), it follows from (12.1.112) and (12.1.113) that

$$\alpha_{i,\varepsilon} = o\left(\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx\right) \tag{12.1.120}$$

Then, by (12.1.110) and (12.1.120),

$$\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx \geq o\left(\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx\right) + \mu_{k_0+1,\varepsilon}\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}R_\varepsilon^2 dx \tag{12.1.121}$$

Independently, it is easily seen that

$$\begin{aligned}\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}w_\varepsilon^2 dx &= \sum_{i=1}^{k}\alpha_{i,\varepsilon}^2 + \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}R_\varepsilon^2 dx \\ &= o\left(\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx\right) + \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}R_\varepsilon^2 dx\end{aligned} \tag{12.1.122}$$

Then, it follows from (12.1.121) and (12.1.122) that

$$\liminf_{\varepsilon\to 0} \frac{\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx}{\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}w_\varepsilon^2 dx} \geq \mu_{k_0+1} \tag{12.1.123}$$

and since $\mu_{k_0+1} > \frac{2^\star-1}{K_n}$, (12.1.123) is in contradiction with (12.1.97). In particular, (12.1.95) is proved.

The final argument in the proof of (12.1.4) and (12.1.5) goes as follows. Combining (12.1.94) and (12.1.95) we get that

$$\lim_{\varepsilon\to 0} \varepsilon\hat{\mu}_\varepsilon^{2-n} = \frac{(n-4)\omega_{n-1}}{4n(n-2)}A_n^2 \tag{12.1.124}$$

By (12.1.42),

$$\alpha\lim_{\varepsilon\to 0}\hat{\mu}_\varepsilon^{4-n}r_\varepsilon^2 = \frac{(n-4)\omega_{n-1}A_n^2}{16n(n-1)} \tag{12.1.125}$$

when $n \geq 5$, and by (12.1.56),

$$\lim_{\varepsilon\to 0} B_\varepsilon r_\varepsilon^{n+2} = \frac{2n(n+2)}{\omega_{n-1}} \tag{12.1.126}$$

It follows from (12.1.124)-(12.1.126) that

$$\begin{aligned}&\lim_{\varepsilon\to 0} B_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} \\ &= \frac{2n(n+2)\omega_n^{2(n+2)/n}}{\omega_{n-1}^{2n/(n-2)}}\left(4^{n-3}n(n-2)(n-4)\right)^{-\frac{n+2}{n-2}}\left(\frac{4(n-1)}{n-2}\alpha\right)^{\frac{n+2}{2}}\end{aligned}$$

when $n \geq 5$. This proves (12.1.5). We need some more work to get (12.1.4). Assuming that $n = 4$, it follows from (12.1.69) that $\varepsilon = O(\hat{\mu}_\varepsilon^2)$. Hence,

$$\limsup_{\varepsilon \to 0} \frac{|\ln \hat{\mu}_\varepsilon|}{|\ln \varepsilon|} \leq \frac{1}{2} \tag{12.1.127}$$

By (12.1.43) and (12.1.56),

$$\begin{aligned} &\lim_{\varepsilon \to 0} |\ln \hat{\mu}_\varepsilon| r_\varepsilon^2 = \frac{4}{\alpha} \\ &\lim_{\varepsilon \to 0} B_\varepsilon r_\varepsilon^6 = \frac{48}{\omega_3} \end{aligned} \tag{12.1.128}$$

Writing that

$$\frac{B_\varepsilon}{|\ln \varepsilon|^3} = \frac{B_\varepsilon r_\varepsilon^6}{(r_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|)^3} \left(\frac{|\ln \hat{\mu}_\varepsilon|}{|\ln \varepsilon|} \right)^3$$

it follows from (12.1.127) and (12.1.128) that

$$\limsup_{\varepsilon \to 0} \frac{B_\varepsilon}{|\ln \varepsilon|^3} \leq \frac{3\alpha^3}{32\omega_3} \tag{12.1.129}$$

Conversely, we claim that

$$\liminf_{\varepsilon \to 0} \frac{B_\varepsilon}{|\ln \varepsilon|^3} \geq \frac{3\alpha^3}{32\omega_3} \tag{12.1.130}$$

For that purpose, we let f_ε be the function given by

$$f_\varepsilon(x) = \frac{\lambda_\varepsilon}{\lambda_\varepsilon^2 + \frac{\sqrt{\omega_4}}{4}|x|^2} + a_\varepsilon \lambda_\varepsilon \left(|x|^2 - b_\varepsilon \right) \tag{12.1.131}$$

where λ_ε, a_ε, and b_ε are real numbers. Given $k_\varepsilon > 0$, we let also $\tilde{f}_\varepsilon$ be the function given by

$$\begin{aligned} &\tilde{f}_\varepsilon(x) = \frac{1}{k_\varepsilon} f_\varepsilon \left(\frac{1}{k_\varepsilon} x \right) \quad \text{in } \mathcal{B}_0(k_\varepsilon) \\ &\tilde{f}_\varepsilon(x) = 0 \quad \text{in } \mathcal{B} \backslash \mathcal{B}_0(k_\varepsilon) \end{aligned} \tag{12.1.132}$$

We choose k_ε such that

$$k_\varepsilon^2 = \frac{8}{\alpha |\ln \varepsilon|} \tag{12.1.133}$$

and $\lambda_\varepsilon > 0$ small such that

$$\varepsilon = \frac{\lambda_\varepsilon^2}{|\ln \lambda_\varepsilon|} \tag{12.1.134}$$

Moreover, we choose a_ε and b_ε such that $\tilde{f}_\varepsilon$ is C^1 in $\mathcal{B}$, and hence such that

$$a_\varepsilon = \frac{\sqrt{\omega_4}}{4\left(\lambda_\varepsilon^2 + \frac{\sqrt{\omega_4}}{4} \right)^2} \quad \text{and} \quad b_\varepsilon = \frac{1}{a_\varepsilon \left(\lambda_\varepsilon^2 + \frac{\sqrt{\omega_4}}{4} \right)} + 1 \tag{12.1.135}$$

In particular,

$$
\begin{aligned}
a_\varepsilon &\to \frac{4}{\sqrt{\omega_4}} \\
b_\varepsilon &\to 2
\end{aligned}
\tag{12.1.136}
$$

as $\varepsilon \to 0$. Noting that $f_\varepsilon \geq 0$ in $\mathcal{B}$, we write now that for any $\varepsilon > 0$,

$$
\int_{\mathcal{B}} |\nabla \tilde{f}_\varepsilon|^2 dx - \alpha \int_{\mathcal{B}} \tilde{f}_\varepsilon^2 dx + B_\varepsilon \left(\int_{\mathcal{B}} \tilde{f}_\varepsilon dx \right)^2 \geq \frac{1-\varepsilon}{K_4^2} \left(\int_{\mathcal{B}} \tilde{f}_\varepsilon^4 dx \right)^{1/2} \tag{12.1.137}
$$

Easy computations give that

$$
\int_{\mathcal{B}} \tilde{f}_\varepsilon^2 dx = \frac{16\omega_3}{\omega_4} k_\varepsilon^2 \lambda_\varepsilon^2 |\ln \lambda_\varepsilon| + o(k_\varepsilon^2 \lambda_\varepsilon^2 |\ln \lambda_\varepsilon|) \tag{12.1.138}
$$

and, thanks to (12.1.136), that

$$
\int_{\mathcal{B}} \tilde{f}_\varepsilon dx = \frac{2\omega_3 k_\varepsilon^3 \lambda_\varepsilon}{3\sqrt{\omega_4}} \left(1 + o(1)\right) \tag{12.1.139}
$$

Similarly, we find with (12.1.136) that

$$
\int_{\mathcal{B}} |\nabla \tilde{f}_\varepsilon|^2 dx = \frac{1}{K_4^2} - \frac{256\omega_3}{3\omega_4} \lambda_\varepsilon^2 + o(\lambda_\varepsilon^2) \tag{12.1.140}
$$

and that

$$
\left(\int_{\mathcal{B}} \tilde{f}_\varepsilon^4 dx \right)^{1/2} = 1 - \frac{128\omega_3 K_4^2}{\omega_4} \lambda_\varepsilon^2 + o(\lambda_\varepsilon^2) \tag{12.1.141}
$$

Plugging (12.1.138)-(12.1.141) into (12.1.137), it follows that

$$
\begin{aligned}
&\frac{128\omega_3}{3\omega_4} \lambda_\varepsilon^2 + \frac{4\omega_3^2}{9\omega_4} B_\varepsilon \left(1 + o(1)\right) k_\varepsilon^6 \lambda_\varepsilon^2 + \frac{\varepsilon}{K_4^2} \\
&\quad \geq \frac{16\alpha\omega_3}{\omega_4} k_\varepsilon^2 \lambda_\varepsilon^2 |\ln \lambda_\varepsilon| + o(k_\varepsilon^2 \lambda_\varepsilon^2 |\ln \lambda_\varepsilon|) + o(\lambda_\varepsilon^2)
\end{aligned}
\tag{12.1.142}
$$

By (12.1.133) and (12.1.134), $\varepsilon = o(\lambda_\varepsilon^2)$ and $k_\varepsilon^2 |\ln \lambda_\varepsilon| = \frac{4}{\alpha} + o(1)$. We then get with (12.1.142) that

$$
\frac{B_\varepsilon}{|\ln \varepsilon|^3} \geq \frac{3\alpha^3}{32\omega_3} + o(1)
$$

and (12.1.130) is proved. Then, thanks to (12.1.129) and (12.1.130),

$$
\lim_{\varepsilon \to 0} \frac{B_\varepsilon}{|\ln \varepsilon|^3} = \frac{3\alpha^3}{32\omega_3}
$$

and (12.1.4) is also proved.

12.2 A test function type argument

We let (M, g) be a smooth compact Riemannian manifold, $n \geq 4$, whose scalar curvature is positive somewhere. We let also $x_0 \in M$ be such that

$$S_g(x_0) = \max_{x \in M} S_g(x)$$

where S_g is the scalar curvature of g. For $\delta > 0$ small, we consider $\mathcal{B}_0(\delta)$ the Euclidean ball of center 0 and radius δ, and we still denote by g the metric $\exp_{x_0}^\star g$. Let us assume that for any $u \in C_c^\infty(\mathcal{B}_0(\delta))$,

$$\int_{\mathcal{B}_0(\delta)} |\nabla u|^2 dv_g + \hat{B}_\varepsilon \left(\int_{\mathcal{B}_0(\delta)} |u| dv_g \right)^2 \geq \frac{1-\varepsilon}{K_n} \left(\int_{\mathcal{B}_0(\delta)} |u|^{2^\star} dv_g \right)^{2/2^\star} \tag{12.2.1}$$

The goal in this subsection is to prove that

$$\liminf_{\varepsilon \to 0} \frac{\hat{B}_\varepsilon}{|\ln \varepsilon|^3} \geq \frac{1}{2304\omega_3} \left(\max_{x \in M} S_g \right)^3 \tag{12.2.2}$$

when $n = 4$, and that

$$\liminf_{\varepsilon \to 0} \hat{B}_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} \geq C_n \left(\max_{x \in M} S_g \right)^{\frac{n+2}{2}} \tag{12.2.3}$$

when $n \geq 5$, where

$$C_n = \frac{2n(n+2)\omega_n^{2+\frac{4}{n}}}{\omega_{n-1}^{\frac{2n}{n-2}} \left(4^{n-3} n(n-2)(n-4)\right)^{\frac{n+2}{n-2}}}$$

is as in subsection 12.1. For that purpose, we let B_ε and the u_ε's be as in subsection 12.1, where α is given by

$$\alpha = \frac{n-2}{4(n-1)} S_g(0) \delta^2$$

Then,

$$\begin{aligned} &\Delta u_\varepsilon - \alpha u_\varepsilon + B_\varepsilon \|u_\varepsilon\|_1 \Sigma_\varepsilon = \frac{1-\varepsilon}{K_n} u_\varepsilon^{2^\star - 1} \text{ in } \mathcal{B} \\ &u_\varepsilon = 0 \text{ on } \partial \mathcal{B} , \int_{\mathcal{B}} u_\varepsilon^{2^\star} dx = 1 \end{aligned} \tag{12.2.4}$$

We let z_ε be the function in $\mathcal{B}_0(\delta)$ given by

$$z_\varepsilon(x) = \delta^{1-\frac{n}{2}} u_\varepsilon(\frac{1}{\delta} x) \tag{12.2.5}$$

Then,

$$\int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 dv_g + \hat{B}_\varepsilon \left(\int_{\mathcal{B}_0(\delta)} z_\varepsilon dv_g \right)^2 \geq \frac{1-\varepsilon}{K_n} \left(\int_{\mathcal{B}_0(\delta)} z_\varepsilon^{2^\star} dv_g \right)^{2/2^\star} \tag{12.2.6}$$

Thanks to the Cartan expansion of a metric in geodesic normal coordinates,

$$\begin{aligned} \int_{\mathcal{B}_0(\delta)} z_\varepsilon dv_g &= \int_{\mathcal{B}_0(\delta)} z_\varepsilon dx + O\left(\int_{\mathcal{B}_0(\delta)} |x|^2 z_\varepsilon dx \right) \\ &= \delta^{\frac{n}{2}+1} \int_{\mathcal{B}} u_\varepsilon dx + \delta^{\frac{n}{2}+3} O\left(\int_{\mathcal{B}} |x|^2 u_\varepsilon dx \right) \end{aligned}$$

We have seen in subsection 12.1 that u_ε has its support in $\mathcal{B}_0(r_\varepsilon)$ where $r_\varepsilon \to 0$ as $\varepsilon \to 0$. Hence we can write that

$$\left(\int_{\mathcal{B}_0(\delta)} z_\varepsilon dv_g\right)^2 = \delta^{n+2}\left(1+o(1)\right)\left(\int_{\mathcal{B}} u_\varepsilon dx\right)^2 \tag{12.2.7}$$

Still thanks to the Cartan expansion of a metric in geodesic normal coordinates, and since z_ε is radially symmetrical,

$$\begin{aligned}\int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 dv_g = \int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 dx - \frac{1}{6}R_{ij}(0)\int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 x^i x^j dx\\ +O\left(\int_{\mathcal{B}_0(\delta)} |x|^4 |\nabla z_\varepsilon|^2 dx\right)\end{aligned}$$

where R stands for the Ricci curvature of g. By (12.2.4) we get that

$$\begin{aligned}\int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 dx &= \int_{\mathcal{B}} |\nabla u_\varepsilon|^2 dx\\ &= \frac{1-\varepsilon}{K_n} - B_\varepsilon\left(\int_{\mathcal{B}} u_\varepsilon dx\right)^2 + \alpha\int_{\mathcal{B}} u_\varepsilon^2 dx\end{aligned}$$

Independently, since u_ε is radially symmetrical,

$$\int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 x^i x^j dx = \delta^2\int_{\mathcal{B}} |\nabla u_\varepsilon|^2 x^i x^j dx = \delta^2\frac{1}{n}\delta^{ij}\int_{\mathcal{B}} |x|^2|\nabla u_\varepsilon|^2 dx$$

Noting that u_ε has its support in $\mathcal{B}_0(r_\varepsilon)$, where $r_\varepsilon \to 0$ as $\varepsilon \to 0$, we also have that

$$\int_{\mathcal{B}} |x|^4|\nabla z_\varepsilon|^2 dx = \delta^4\int_{\mathcal{B}} |x|^4|\nabla u_\varepsilon|^2 dx = o\left(\int_{\mathcal{B}} |x|^2|\nabla u_\varepsilon|^2 dx\right)$$

Hence,

$$\begin{aligned}\int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 dv_g = \frac{1-\varepsilon}{K_n} - B_\varepsilon\left(\int_{\mathcal{B}} u_\varepsilon dx\right)^2 + \alpha\int_{\mathcal{B}} u_\varepsilon^2 dx\\ -\frac{\delta^2}{6n}S_g(0)\int_{\mathcal{B}} |x|^2|\nabla u_\varepsilon|^2 dx + o\left(\int_{\mathcal{B}} |x|^2|\nabla u_\varepsilon|^2 dx\right)\end{aligned} \tag{12.2.8}$$

Similar arguments give that

$$\int_{\mathcal{B}_0(\delta)} z_\varepsilon^{2^\star} dv_g = 1 - \frac{\delta^2}{6n}S_g(0)\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx + o\left(\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx\right)$$

and hence that

$$\left(\int_{\mathcal{B}_0(\delta)} z_\varepsilon^{2^\star} dv_g\right)^{2/2^\star} = 1 - \frac{(n-2)\delta^2}{6n^2}S_g(0)\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx + o\left(\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx\right) \tag{12.2.9}$$

Coming back to (12.2.6), we get with (12.2.7)-(12.2.9) that

$$\begin{aligned}\alpha\int_{\mathcal{B}} u_\varepsilon^2 dx + \left(\hat{B}_\varepsilon\left(1+o(1)\right)\delta^{n+2} - B_\varepsilon\right)\left(\int_{\mathcal{B}} u_\varepsilon dx\right)^2\\ -\frac{\delta^2}{6n}S_g(0)\int_{\mathcal{B}} |x|^2|\nabla u_\varepsilon|^2 dx + o\left(\int_{\mathcal{B}} |x|^2|\nabla u_\varepsilon|^2 dx\right)\\ \geq -\frac{(n-2)\delta^2}{6n^2 K_n}S_g(0)\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx + o\left(\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx\right)\end{aligned} \tag{12.2.10}$$

With the notations of subsection 12.1,

$$\left(\int_{\mathcal{B}} u_\varepsilon dx\right)^2 = \frac{A_n^2\omega_{n-1}^2}{4n^2(n+2)^2}\hat{\mu}_\varepsilon^{n-2} r_\varepsilon^{n+2} + o\left(\hat{\mu}_\varepsilon^{n-2} r_\varepsilon^{n+2}\right) \tag{12.2.11}$$

We also have that

$$\int_{\mathcal{B}} u_\varepsilon^2 dx = r_\varepsilon^2 \int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx$$

Hence, by (12.1.41) and (12.1.55),

$$\int_{\mathcal{B}} u_\varepsilon^2 dx = \frac{16\omega_3}{\omega_4} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o\left(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|\right) \tag{12.2.12}$$

if $n = 4$, and

$$\int_{\mathcal{B}} u_\varepsilon^2 dx = \frac{4(n-1)}{n-4} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o\left(r_\varepsilon^2 \hat{\mu}_\varepsilon^2\right) \tag{12.2.13}$$

if $n \geq 5$. Similarly,

$$\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx = r_\varepsilon^2 \int_{\mathcal{B}} |x|^2 \hat{u}_\varepsilon^{2^\star} dx = r_\varepsilon^2 \hat{\mu}_\varepsilon^2 \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} |x|^2 \left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1} \hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)\right)^{2^\star} dx$$

and thanks to (12.1.32) and (12.1.33), we have that

$$\int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} |x|^2 \left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1} \hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)\right)^{2^\star} dx \to \int_{\mathbb{R}^n} |x|^2 V_0^{2^\star} dx$$

as $\varepsilon \to 0$. It is easily seen, see for instance Demengel and Hebey [11], that

$$\int_{\mathbb{R}^n} |x|^2 V_0^{2^\star} dx = 2^{n+1} \omega_n^{-\frac{n+2}{n}} \omega_{n-1} \frac{\Gamma(\frac{n+2}{2})\Gamma(\frac{n-2}{2})}{\Gamma(n)}$$

and since

$$\Gamma(n) = \frac{2^{n-1}\omega_{n-1}}{\omega_n} \Gamma(\frac{n}{2})^2$$

we have that

$$\int_{\mathbb{R}^n} |x|^2 V_0^{2^\star} dx = \frac{4n}{(n-2)\omega_n^{2/n}}$$

Hence,

$$\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx = \frac{4n}{(n-2)\omega_n^{2/n}} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o\left(r_\varepsilon^2 \hat{\mu}_\varepsilon^2\right) \tag{12.2.14}$$

Integrating by parts, and thanks to (12.2.4), we also have that

$$\begin{aligned}
&\int_{\mathcal{B}} |x|^2 |\nabla u_\varepsilon|^2 dx = n \int_{\mathcal{B}} u_\varepsilon^2 dx + \int_{\mathcal{B}} |x|^2 u_\varepsilon \Delta u_\varepsilon dx \\
&= n \int_{\mathcal{B}} u_\varepsilon^2 dx + \frac{1-\varepsilon}{K_n} \int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx - B_\varepsilon \|u_\varepsilon\|_1 \int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx + \alpha \int_{\mathcal{B}} |x|^2 u_\varepsilon^2 dx
\end{aligned}$$

Noting that

$$\int_{\mathcal{B}} |x|^2 u_\varepsilon dx = r_\varepsilon^2 O\left(\int_{\mathcal{B}} u_\varepsilon dx\right) \quad \text{and} \quad \int_{\mathcal{B}} |x|^2 u_\varepsilon^2 dx = o\left(\int_{\mathcal{B}} u_\varepsilon^2 dx\right)$$

it follows that

$$\int_{\mathcal{B}} |x|^2 |\nabla u_\varepsilon|^2 dx = \frac{64\omega_3}{\omega_4} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \mu_\varepsilon| + o\left(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \mu_\varepsilon|\right) + O\left(B_\varepsilon r_\varepsilon^8 \hat{\mu}_\varepsilon^2\right) \tag{12.2.15}$$

when $n = 4$, and

$$\int_{\mathcal{B}} |x|^2 |\nabla u_\varepsilon|^2 dx = \frac{n(n^2-4)}{n-4} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o\left(r_\varepsilon^2 \hat{\mu}_\varepsilon^2\right) + O\left(B_\varepsilon r_\varepsilon^{n+4} \hat{\mu}_\varepsilon^{n-2}\right) \tag{12.2.16}$$

when $n \geq 5$. Let us assume first that $n \geq 5$. Plugging (12.2.11)-(12.2.16) into (12.2.10), and thanks to the choice we made for α, we get that

$$\begin{aligned} &\left(\hat{B}_\varepsilon \delta^{n+2} - B_\varepsilon\right) \frac{A_n^2 \omega_{n-1}^2}{4n^2(n+2)^2} r_\varepsilon^{n+2} \hat{\mu}_\varepsilon^{n-2} \\ &\quad \geq o\left(r_\varepsilon^2 \hat{\mu}_\varepsilon^2\right) + o\left(\hat{B}_\varepsilon r_\varepsilon^{n+2} \hat{\mu}_\varepsilon^{n-2}\right) + O\left(B_\varepsilon r_\varepsilon^{n+4} \hat{\mu}_\varepsilon^{n-2}\right) \end{aligned}$$

so that

$$\left(\frac{\hat{B}_\varepsilon}{B_\varepsilon} \delta^{n+2} - 1\right) \frac{A_n^2 \omega_{n-1}^2}{4n^2(n+2)^2} \geq o\left(r_\varepsilon^{-n} \hat{\mu}_\varepsilon^{4-n} B_\varepsilon^{-1}\right) + o\left(\frac{\hat{B}_\varepsilon}{B_\varepsilon}\right) + O\left(r_\varepsilon^2\right) \tag{12.2.17}$$

By (12.1.42) and (12.1.56),

$$r_\varepsilon^{-n} \hat{\mu}_\varepsilon^{4-n} B_\varepsilon^{-1} = O(1)$$

Hence, (12.2.17) gives that

$$\liminf_{\varepsilon \to 0} \frac{\hat{B}_\varepsilon}{B_\varepsilon} \delta^{n+2} \geq 1$$

and thanks to (12.1.5) we get that

$$\liminf_{\varepsilon \to 0} \hat{B}_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} \geq C_n S_g(0)^{\frac{n+2}{2}}$$

where

$$C_n = \frac{2n(n+2)\omega_n^{2+\frac{4}{n}}}{\omega_{n-1}^{\frac{2n}{n-2}} \left(4^{n-3} n(n-2)(n-4)\right)^{\frac{n+2}{n-2}}}$$

This proves (12.2.3). Let us assume now that $n = 4$. Plugging (12.2.11)-(12.2.16) into (12.2.10), and thanks to the choice we made for α, we get that

$$\frac{\hat{B}_\varepsilon}{B_\varepsilon} \delta^6 - 1 \geq o\left(\frac{\hat{B}_\varepsilon}{B_\varepsilon}\right) + o\left(B_\varepsilon^{-1} r_\varepsilon^{-4} |\ln \hat{\mu}_\varepsilon|\right) + O\left(r_\varepsilon^2\right) \tag{12.2.18}$$

By (12.1.43) and (12.1.56),

$$B_\varepsilon^{-1} r_\varepsilon^{-4} |\ln \hat{\mu}_\varepsilon| = O(1)$$

Hence, (12.2.18) gives that

$$\liminf_{\varepsilon\to 0}\frac{\hat{B}_\varepsilon}{B_\varepsilon}\delta^6 \geq 1$$

and thanks to (12.1.4) we get that

$$\liminf_{\varepsilon\to 0}\frac{\hat{B}_\varepsilon}{|\ln\varepsilon|^3} \geq \frac{1}{2304\omega_3}S_g(0)^3$$

This proves (12.2.4).

12.3 The Riemannian case

As in subsection 12.2, we let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 4$. We assume that the scalar curvature S_g of g is such that $\max_{x\in M} S_g > 0$. For $\varepsilon > 0$ small, we let $\hat{B}_\varepsilon$ be the smallest B such that for all $u \in C^\infty(M)$,

$$\frac{1-\varepsilon}{K_n}\|u\|_{2^\star}^2 \leq \|\nabla u\|_2^2 + B\|u\|_1^2$$

As in subsection 12.1, it can be proved that

$$\inf_{u\in C^\infty(M)\setminus\{0\}}\frac{\|\nabla u\|_2^2 + \hat{B}_\varepsilon\|u\|_1^2}{\|u\|_{2^\star}^2} = \frac{1-\varepsilon}{K_n} \tag{12.3.1}$$

With respect to the notations of the introduction, we have that

$$\hat{B}_\varepsilon = \frac{1-\varepsilon}{K_n}B_{\frac{K_n\varepsilon}{1-\varepsilon}}(g) \quad \text{and} \quad B_\varepsilon(g) = (K_n+\varepsilon)\,\hat{B}_{\frac{\varepsilon}{K_n+\varepsilon}} \tag{12.3.2}$$

The goal in this section is to prove that

$$\limsup_{\varepsilon\to 0}\frac{\hat{B}_\varepsilon}{|\ln\varepsilon|^3} \leq \frac{1}{2304\omega_3}\left(\max_{x\in M} S_g\right)^3 \tag{12.3.3}$$

when $n = 4$, and that

$$\limsup_{\varepsilon\to 0}\hat{B}_\varepsilon\varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} \leq C_n\left(\max_{x\in M} S_g\right)^{\frac{n+2}{2}} \tag{12.3.4}$$

when $n \geq 5$, where

$$C_n = \frac{2n(n+2)\omega_n^{2+\frac{4}{n}}}{\omega_{n-1}^{\frac{2n}{n-2}}\left(4^{n-3}n(n-2)(n-4)\right)^{\frac{n+2}{n-2}}}$$

is as in subsections 12.1 and 12.2. As indicated at the end of this section, the second part Theorem 4.4 follows from (12.3.3), (12.3.4), and what is proved in subsection 12.2.

Thanks to Theorem 4.1, $\hat{B}_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Independently, the LHS in (12.3.1) being less than $1/K_n$, it easily follows from standard variational arguments, as in subsection 12.1, that there exists a minimizer for the infimum in (12.3.1). With no risk of confusion with the notations of subsection 12.1, we denote by u_ε this minimizer. We then get that for any $\varepsilon > 0$, there exists $u_\varepsilon \in C^{1,\beta}(M)$, $0 < \beta < 1$, such that

$$\Delta_g u_\varepsilon + \hat{B}_\varepsilon \|u_\varepsilon\|_1 \Sigma_\varepsilon = \frac{1-\varepsilon}{K_n} u_\varepsilon^{2^\star - 1} \tag{12.3.5}$$

and

$$\int_M u_\varepsilon^{2^\star} dv_g = 1\,, \quad u_\varepsilon \geq 0 \text{ in } M \tag{12.3.6}$$

where $\Delta_g = -div_g(\nabla)$ is the Riemannian Laplacian, and $\Sigma_\varepsilon \in L^\infty(M)$, $0 \leq \Sigma_\varepsilon \leq 1$, is such that $\Sigma_\varepsilon u_\varepsilon = u_\varepsilon$. We let x_ε be a point where u_ε is maximum, and set

$$\mu_\varepsilon^{1-\frac{n}{2}} = \|u_\varepsilon\|_\infty = u_\varepsilon(x_\varepsilon) \tag{12.3.7}$$

Multiplying (12.3.5) by u_ε and integrating over M, we get with (12.3.6) that

$$\hat{B}_\varepsilon \|u_\varepsilon\|_1^2 \leq \frac{1}{K_n}$$

Since $\hat{B}_\varepsilon \to +\infty$ as $\varepsilon \to 0$, it follows that $\|u_\varepsilon\|_1 \to 0$ as $\varepsilon \to 0$. In particular, by Hölder's inequality and (12.3.6), $\|u_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0$. Noting that

$$1 = \int_M u_\varepsilon^{2^\star} dv_g \leq \mu_\varepsilon^{-\frac{n+2}{2}} \int_M u_\varepsilon dv_g$$

we also have that

$$\lim_{\varepsilon \to 0} \mu_\varepsilon = 0 \tag{12.3.8}$$

Independently, by Hebey and Vaugon [27], there exists $B > 0$ such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \leq K_n \|\nabla u\|_2^2 + B\|u\|_2^2$$

Taking $u = u_\varepsilon$ in this inequality,

$$1 - B\|u_\varepsilon\|_2^2 \leq K_n \|\nabla u_\varepsilon\|_2^2 = 1 - \varepsilon - K_n \hat{B}_\varepsilon \|u_\varepsilon\|_1^2$$

and it follows that

$$\lim_{\varepsilon \to 0} \hat{B}_\varepsilon \|u_\varepsilon\|_1^2 = 0 \tag{12.3.9}$$

As in section 8, there exists $x_0 \in M$ such that for any $\delta > 0$,

$$\lim_{\varepsilon \to 0} \int_{B_{x_0}(\delta)} u_\varepsilon^{2^\star} dv_g = 1$$

and

$$u_\varepsilon \to 0 \text{ in } C^0_{loc}(M\backslash\{x_0\}) \tag{12.3.10}$$

as ε goes to 0. According to what we just said, $x_\varepsilon \to x_0$ and $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. By (12.3.9), noting that

$$1 = \|u_\varepsilon\|_{2^\star}^{2^\star} \leq \|u_\varepsilon\|_\infty^{2^\star - 1} \|u_\varepsilon\|_1$$

we get that

$$\lim_{\varepsilon \to 0} \hat{B}_\varepsilon \mu_\varepsilon^{\frac{n+2}{2}} \|u_\varepsilon\|_1 = 0 \tag{12.3.11}$$

In what follows we let $\exp_{x_\varepsilon}$ be the exponential map at x_ε. There clearly exists $\delta > 0$, independent of ε, such that for any ε, $\exp_{x_\varepsilon}$ is a diffeomorphism from $\mathcal{B}_0(\delta) \subset I\!R^n$ onto $B_{x_\varepsilon}(\delta)$. As a starting point in the proof of (12.3.3) and (12.3.4), we prove weak estimates on the u_ε's.

12.3.1 Weak Estimates

For $x \in \mathcal{B}_0(\mu_\varepsilon^{-1}\delta)$, we set

$$\tilde{g}_\varepsilon(x) = \left(\exp_{x_\varepsilon}^\star g\right)(\mu_\varepsilon x) \quad , \quad \tilde{u}_\varepsilon(x) = \mu_\varepsilon^{\frac{n}{2}-1} u_\varepsilon\left(\exp_{x_\varepsilon}(\mu_\varepsilon x)\right)$$

and $\tilde{\Sigma}_\varepsilon(x) = \Sigma_\varepsilon\left(\exp_{x_\varepsilon}(\mu_\varepsilon x)\right)$. It is easily seen that

$$\Delta_{\tilde{g}_\varepsilon}\tilde{u}_\varepsilon + \hat{B}_\varepsilon \mu_\varepsilon^{\frac{n+2}{2}}\|u_\varepsilon\|_1 \tilde{\Sigma}_\varepsilon = \frac{1-\varepsilon}{K_n}\tilde{u}_\varepsilon^{2^\star-1} \tag{12.3.12}$$

Moreover,

$$\tilde{u}_\varepsilon(0) = \|\tilde{u}_\varepsilon\|_\infty = 1 \tag{12.3.13}$$

and if ξ stands for the Euclidean metric of $I\!R^n$,

$$\lim_{\varepsilon\to 0}\tilde{g}_\varepsilon = \xi \quad \text{in } C^2(K) \tag{12.3.14}$$

for any compact subset K of $I\!R^n$. Thanks to (12.3.11)-(12.3.14), we get by standard elliptic theory, as developed in Gilbarg-Trudinger [22], that there exists some $\tilde{u} \in C^1(I\!R^n)$ such that for any compact subset K of $I\!R^n$,

$$\lim_{\varepsilon\to 0}\tilde{u}_\varepsilon = \tilde{u} \quad \text{in } C^1(K) \tag{12.3.15}$$

Clearly, $\tilde{u}(0) = 1$ and $\tilde{u} \not\equiv 0$. Moreover, it is easily seen that $\tilde{u} \in D_1^2(I\!R^n)$, where $D_1^2(I\!R^n)$ is the homogeneous Euclidean Sobolev space. By passing to the limit as ε goes to 0 in (12.3.12), according to (12.3.11), (12.3.14), and (12.3.15), we get that $\tilde{u}$ is a solution of

$$\Delta\tilde{u} = \frac{1}{K_n}\tilde{u}^{2^\star-1}$$

By Caffarelli-Gidas-Spruck [8], and also Obata [32],

$$\tilde{u}(x) = \left(1 + \frac{\omega_n^{2/n}}{4}|x|^2\right)^{1-\frac{n}{2}} \tag{12.3.16}$$

Noting that $\tilde{u}$ is of norm 1 in $L^{2^\star}(I\!R^n)$, and that for any $R > 0$,

$$\int_{B_{x_\varepsilon}(R\mu_\varepsilon)} u_\varepsilon^{2^\star}\,dv_g = \int_{\mathcal{B}_0(R)}\tilde{u}_\varepsilon^{2^\star}\,dv_{\tilde{g}_\varepsilon}$$

we get that

$$\lim_{\varepsilon\to 0}\int_{B_{x_\varepsilon}(R\mu_\varepsilon)} u_\varepsilon^{2^\star}\,dv_g = 1 - \int_{I\!R^n\backslash\mathcal{B}_0(R)}\tilde{u}^{2^\star}\,dx$$

Hence,

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{B_{x_\varepsilon}(R\mu_\varepsilon)} u_\varepsilon^{2^\star}\,dv_g = 1 \tag{12.3.17}$$

We claim now that the two following estimates hold. On the one hand that there exists $C > 0$, such that for any ε, and any x,

$$d_g(x_\varepsilon, x)^{\frac{n}{2}-1}u_\varepsilon(x) \le C \tag{12.3.18}$$

where d_g is the distance with respect to g. On the other hand that

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\sup_{x\in M\backslash B_{x_\varepsilon}(R\mu_\varepsilon)} d_g(x_\varepsilon,x)^{\frac{n}{2}-1}u_\varepsilon(x)=0 \tag{12.3.19}$$

In order to prove (12.3.18), we set

$$v_\varepsilon(x)=d_g(x_\varepsilon,x)^{\frac{n}{2}-1}u_\varepsilon(x)$$

and assume by contradiction that for some subsequence,

$$\lim_{\varepsilon\to 0}\|v_\varepsilon\|_\infty=+\infty \tag{12.3.20}$$

Let y_ε be some point in M where v_ε is maximum. By (12.3.10), $y_\varepsilon\to x_0$ as $\varepsilon\to 0$, while by (12.3.20),

$$\lim_{\varepsilon\to 0}\frac{d_g(x_\varepsilon,y_\varepsilon)}{\mu_\varepsilon}=+\infty \tag{12.3.21}$$

Fix now $\delta>0$ small, and set

$$\Omega_\varepsilon=u_\varepsilon(y_\varepsilon)^{\frac{2}{n-2}}\exp_{y_\varepsilon}^{-1}\left(B_{x_\varepsilon}(\delta)\right)$$

For $x\in\Omega_\varepsilon$, define

$$\tilde v_\varepsilon(x)=u_\varepsilon(y_\varepsilon)^{-1}u_\varepsilon\left(\exp_{y_\varepsilon}(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}x)\right)$$

and

$$h_\varepsilon(x)=\left(\exp_{y_\varepsilon}^\star g\right)\left(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}x\right)$$

It easily follows from (12.3.20), since M is compact, that $u_\varepsilon(y_\varepsilon)\to+\infty$ as $\varepsilon\to 0$. Hence,

$$\lim_{\varepsilon\to 0}h_\varepsilon=\xi\quad\text{in } C^2\left(\mathcal{B}_0(2)\right) \tag{12.3.22}$$

where ξ is the Euclidean metric. Independently, we have that

$$\Delta_{h_\varepsilon}\tilde v_\varepsilon\le\frac{1-\varepsilon}{K_n}\tilde v_\varepsilon^{2^\star-1} \tag{12.3.23}$$

Since $v_\varepsilon(y_\varepsilon)$ goes to $+\infty$, for ε small, and all $x\in\mathcal{B}_0(2)$,

$$d_g\left(x_\varepsilon,\exp_{y_\varepsilon}(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}x)\right)\ge\frac{1}{2}d_g(x_\varepsilon,y_\varepsilon) \tag{12.3.24}$$

This implies that

$$\begin{aligned}\tilde v_\varepsilon(x)&\le 2^{\frac{n}{2}-1}d_g(x_\varepsilon,y_\varepsilon)^{1-\frac{n}{2}}u_\varepsilon(y_\varepsilon)^{-1}v_\varepsilon\left(\exp_{y_\varepsilon}(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}x)\right)\\&\le 2^{\frac{n}{2}-1}d_g(x_\varepsilon,y_\varepsilon)^{1-\frac{n}{2}}u_\varepsilon(y_\varepsilon)^{-1}v_\varepsilon(y_\varepsilon)\end{aligned}$$

so that for ε small,

$$\sup_{x\in\mathcal{B}_0(2)}\tilde v_\varepsilon(x)\le 2^{\frac{n}{2}-1} \tag{12.3.25}$$

By (12.3.21) and (12.3.24), given $R > 0$, and for ε small,

$$B_{y_\varepsilon}(2u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}) \bigcap B_{x_\varepsilon}(R\mu_\varepsilon) = \emptyset \tag{12.3.26}$$

Noting that

$$\int_{\mathcal{B}_0(2)} \tilde{v}_\varepsilon^{2^\star} dv_{h_\varepsilon} = \int_{B_{y_\varepsilon}(2u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}})} u_\varepsilon^{2^\star} dv_g$$

it follows from (12.3.17) and (12.3.26) that

$$\lim_{\varepsilon \to 0} \int_{\mathcal{B}_0(2)} \tilde{v}_\varepsilon^{2^\star} dv_{h_\varepsilon} = 0 \tag{12.3.27}$$

By (12.3.22), (12.3.23), (12.3.25), (12.3.27), and the De Giorgi-Nash-Moser iterative scheme we get that

$$\lim_{\varepsilon \to 0} \sup_{x \in \mathcal{B}_0(1)} \tilde{v}_\varepsilon(x) = 0$$

But $\tilde{v}_\varepsilon(0) = 1$, so that (12.3.20) must be false. This proves (12.3.18). In order to prove (12.3.19), we let v_ε be as above, and proceed once more by contradiction. Then there exists $y_\varepsilon \in M$ and $k_0 > 0$ such that

$$\lim_{\varepsilon \to 0} \frac{d_g(x_\varepsilon, y_\varepsilon)}{\mu_\varepsilon} = +\infty \text{ and } v_\varepsilon(y_\varepsilon) \geq k_0$$

As above, we fix $\delta > 0$ small, and set

$$\Omega_\varepsilon = u_\varepsilon(y_\varepsilon)^{\frac{2}{n-2}} \exp_{y_\varepsilon}^{-1}(B_{x_\varepsilon}(\delta))$$

For $x \in \Omega_\varepsilon$, we define

$$\tilde{v}_\varepsilon(x) = u_\varepsilon(y_\varepsilon)^{-1} u_\varepsilon \left(\exp_{y_\varepsilon}(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}} x)\right)$$

and

$$h_\varepsilon(x) = \left(\exp_{y_\varepsilon}^\star g\right) \left(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}} x\right)$$

Once again

$$\Delta_{h_\varepsilon} \tilde{v}_\varepsilon \leq \frac{1-\varepsilon}{K_n} \tilde{v}_\varepsilon^{2^\star - 1}$$

As when proving (12.3.18), for any $x \in \mathcal{B}_0(\frac{1}{2}k_0^{\frac{2}{n-2}})$,

$$d_g(x_\varepsilon, z_\varepsilon) \geq \frac{1}{2} d_g(x_\varepsilon, y_\varepsilon)$$

and

$$\tilde{v}_\varepsilon(x) = u_\varepsilon(y_\varepsilon)^{-1} v_\varepsilon(z_\varepsilon) d_g(x_\varepsilon, z_\varepsilon)^{1-\frac{n}{2}}$$

where $z_\varepsilon = \exp_{y_\varepsilon}(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}} x)$. It follows from (12.3.18) that

$$\tilde{v}_\varepsilon(x) \leq C 2^{\frac{n}{2}-1} k_0^{-1}$$

Noting that for $R > 0$, and for ε small,

$$B_{y_\varepsilon}\left(\frac{1}{2} k_0^{\frac{2}{n-2}} u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}\right) \bigcap B_{x_\varepsilon}(R\mu_\varepsilon) = \emptyset$$

we conclude as when proving (12.3.18) that (12.3.19) holds.

Now we need stronger estimates than (12.3.18) and (12.3.19). This is the subject of the following subsection.

12.3.2 Strong Estimates.1

In order to get stronger estimates than (12.3.18) and (12.3.19), we let $h_0 \in C^\infty(M)$ be such that $h_0 \geq 0$, $h_0 \not\equiv 0$, $h_0 \equiv 0$ in $B_{x_0}(\delta_0)$. As a remark, it is easy to check that such a choice of h_0 implies that $\Delta_g + h_0$ is coercive. We define L_ε by

$$L_\varepsilon u = \Delta_g u + h_0 u - \frac{1-\varepsilon}{K_n} u_\varepsilon^{2^\star-2} u$$

and claim that L_ε satisfies the maximum principle in $M\backslash B_{x_\varepsilon}(R\mu_\varepsilon)$ for $R > 0$ large and $\varepsilon > 0$ small. Let indeed $z \in C^1\left(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon)\right)$ be such that $z \geq 0$ on $\partial B_{x_\varepsilon}(R\mu_\varepsilon)$ and $L_\varepsilon z \geq 0$. Set $z^- = \max(0, -z)$. Then,

$$\begin{aligned}
0 &\leq \int_{M\backslash B_{x_\varepsilon}(R\mu_\varepsilon)} z^- L_\varepsilon z dv_g \\
&= -\int_{M\backslash B_{x_\varepsilon}(R\mu_\varepsilon)} |\nabla z^-|^2 dv_g - \int_{M\backslash B_{x_\varepsilon}(R\mu_\varepsilon)} h_0 (z^-)^2 dv_g \\
&\quad + \frac{1-\varepsilon}{K_n} \int_{M\backslash B_{x_\varepsilon}(R\mu_\varepsilon)} u_\varepsilon^{2^\star-2} (z^-)^2 dv_g
\end{aligned}$$

while, thanks to Hölder's inequality,

$$\int_{M\backslash B_{x_\varepsilon}(R\mu_\varepsilon)} u_\varepsilon^{2^\star-2}(z^-)^2 dv_g \leq \|u_\varepsilon\|^{2^\star-2}_{L^{2^\star}(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon))} \|z^-\|^2_{L^{2^\star}(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon))}$$

Thus,

$$\begin{aligned}
0 \leq &-\|\nabla z^-\|^2_{L^2(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon))} - \|\sqrt{h_0} z^-\|^2_{L^2(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon))} \\
&+ \frac{1-\varepsilon}{K_n} \|u_\varepsilon\|^{2^\star-2}_{L^{2^\star}(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon))} \|z^-\|^2_{L^{2^\star}(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon))}
\end{aligned} \tag{12.3.28}$$

By (12.3.17),

$$\lim_{R\to+\infty} \lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^{2^\star}(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon))} = 0$$

It follows that for any $A > 0$, there exists $\varepsilon_A > 0$ and $R_A > 0$ such that for $R \geq R_A$ and $\varepsilon \in (0, \varepsilon_A)$,

$$\|u_\varepsilon\|_{L^{2^\star}(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon))} \leq A$$

Let $\lambda > 0$, given by the coercivity of $\Delta_g + h_0$, be such that

$$\lambda \|z^-\|^2_{L^{2^\star}(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon))} \leq \|\nabla z^-\|^2_{L^2(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon))} + \|\sqrt{h_0} z^-\|^2_{L^2(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon))}$$

Coming back to (12.3.28), we get that

$$0 \leq \|z^-\|^2_{L^{2^\star}(M\backslash B_{x_\varepsilon}(R\mu_\varepsilon))} \left(\frac{1-\varepsilon}{K_n} A^{2^\star-2} - \lambda\right)$$

Choosing $A > 0$ small, this implies that $z^- \equiv 0$. The claim is proved. Now, thanks to the De Giorgi-Nash-Moser iterative scheme applied to $\Delta_g u_\varepsilon \leq K_n^{-1} u_\varepsilon^{2^\star-1}$, we have that for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$\sup_{M\backslash B_{x_0}(\delta)} u_\varepsilon \leq C_\delta \|u_\varepsilon\|_1 \tag{12.3.29}$$

Taking $\delta = \delta_0$, this implies that

$$L_\varepsilon u_\varepsilon \leq 0 \quad \text{in } M \tag{12.3.30}$$

We let $\varepsilon_0 > 0$ be such that $\Delta_g + h_0 - \varepsilon_0$ is still coercive in M, and let $G(x, y)$ be the Green function of this operator. We set $H(x) = G(x_\varepsilon, x)$. Given $\nu \in (0, 1)$, we have that

$$\frac{L_\varepsilon H^{1-\nu}}{H^{1-\nu}} = \nu(1-\nu)\frac{|\nabla H|^2}{H^2} + \hat{h}_0 - \frac{1-\varepsilon}{K_n} u_\varepsilon^{2^\star - 2} \tag{12.3.31}$$

where $\hat{h}_0 = (1-\nu)\varepsilon_0 + \nu h_0$. A standard property of the Green function (F.Robert, private communication) is that there exists $\rho > 0$ and $C > 0$ such that for any $x \in B_{x_\varepsilon}(\rho)\backslash\{x_\varepsilon\}$,

$$\frac{|\nabla G(x_\varepsilon, x)|}{G(x_\varepsilon, x)} \geq C \frac{1}{d_g(x_\varepsilon, x)}$$

where d_g is the distance with respect to g. We also have that

$$\left| d_g(x_\varepsilon, x)^{n-2} G(x_\varepsilon, x) \right| \leq C$$

for any $x \neq x_\varepsilon$, where $C > 0$ does not depend on ε, and that

$$d_g(x_\varepsilon, x)^{n-2} G(x_\varepsilon, x) \geq C$$

as soon as $d_g(x_\varepsilon, x) \leq r_0$, where both $r_0 > 0$ and $C > 0$ do not depend on ε. Then, for $x \in B_{x_\varepsilon}(\rho)\backslash\{x_\varepsilon\}$,

$$\frac{L_\varepsilon H^{1-\nu}}{H^{1-\nu}}(x) \geq d_g(x_\varepsilon, x)^{-2} \left(C\nu(1-\nu) - \frac{1-\varepsilon}{K_n} d_g(x_\varepsilon, x)^2 u_\varepsilon^{2^\star-2} \right)$$

and thanks to (12.3.19) we get that for $R > 0$ sufficiently large and $\varepsilon > 0$ sufficiently small,

$$\frac{L_\varepsilon H^{1-\nu}}{H^{1-\nu}} \geq 0 \quad \text{in } B_{x_\varepsilon}(\rho)\backslash B_{x_\varepsilon}(R\mu_\varepsilon) \tag{12.3.32}$$

In $M\backslash B_{x_\varepsilon}(\rho)$, we have (12.3.10). Thus, by (12.3.31), and for $\varepsilon > 0$ small,

$$\frac{L_\varepsilon H^{1-\nu}}{H^{1-\nu}} \geq \hat{h}_0 - \frac{1-\varepsilon}{K_n} u_\varepsilon^{2^\star-2} \geq 0 \quad \text{in } M\backslash B_{x_\varepsilon}(\rho) \tag{12.3.33}$$

Summarizing, it follows from (12.3.30), (12.3.32), and (12.3.33), that there exists $R > 0$, depending on ν, such that

$$L_\varepsilon u_\varepsilon \leq 0 \leq L_\varepsilon H^{1-\nu} \quad \text{in } M\backslash B_{x_\varepsilon}(R\mu_\varepsilon) \tag{12.3.34}$$

By (12.3.15) and (12.3.16), there exists $C > 0$ such that

$$u_\varepsilon \leq C\mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)} H^{1-\nu} \quad \text{on } \partial B_{x_\varepsilon}(R\mu_\varepsilon) \tag{12.3.35}$$

The maximum principle, (12.3.34), and (12.3.35), then give that

$$u_\varepsilon(x) \leq C\mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)} d_g(x_\varepsilon, x)^{(2-n)(1-\nu)}$$

in $M\backslash B_{x_\varepsilon}(R\mu_\varepsilon)$. Noting that this inequality is satisfied in $B_{x_\varepsilon}(R\mu_\varepsilon)$ thanks to (12.3.15) and (12.3.16), we have proved that for any $\nu \in (0,1)$, there exists $C(\nu) > 0$ such that

$$u_\varepsilon(x) \le C\mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)} d_g(x_\varepsilon, x)^{(2-n)(1-\nu)} \tag{12.3.36}$$

for any $\varepsilon > 0$ and any $x \in M\backslash\{x_\varepsilon\}$. Now we claim that there actually exists $C > 0$ such that for any $\varepsilon > 0$ and any $x \in M$,

$$\mu_\varepsilon^{1-\frac{n}{2}} d_g(x_\varepsilon, x)^{n-2} u_\varepsilon(x) \le C \tag{12.3.37}$$

In other words, we claim that (12.3.36) holds with $\nu = 0$. We let $G_0(x,y)$ be the Green function of $\Delta_g + h_0$. Thanks to (12.3.29), noting that $\Sigma_\varepsilon(x) = 1$ if $u_\varepsilon(x) \neq 0$,

$$h_0 u_\varepsilon - \hat{B}_\varepsilon \|u_\varepsilon\|_1 \Sigma_\varepsilon \le 0$$

in M when $\varepsilon > 0$ is small. For (y_ε) a sequence in M, we can then write that for $\varepsilon > 0$ small,

$$\begin{aligned} u_\varepsilon(y_\varepsilon) &= \int_M G_0(x, y_\varepsilon) \left(\Delta_g u_\varepsilon + h_0 u_\varepsilon\right)(x) dv_g(x) \\ &= \frac{1-\varepsilon}{K_n} \int_M G_0(x, y_\varepsilon) u_\varepsilon^{2^\star - 1} dv_g \\ &\quad + \int_M G_0(x, y_\varepsilon) \left(h_0 u_\varepsilon - \hat{B}_\varepsilon \|u_\varepsilon\|_1 \Sigma_\varepsilon\right)(x) dv_g(x) \\ &\le \frac{1}{K_n} \int_M G_0(x, y_\varepsilon) u_\varepsilon^{2^\star - 1} dv_g \end{aligned} \tag{12.3.38}$$

We set

$$\Phi_\varepsilon = u_\varepsilon(y_\varepsilon) \mu_\varepsilon^{1-\frac{n}{2}} d_g(x_\varepsilon, y_\varepsilon)^{n-2}$$

and let H_ε be such that $H_\varepsilon(x) = G_0(x, y_\varepsilon)$. We distinguish three cases.

Case 1: we assume that $\mu_\varepsilon^{-1} d_g(x_\varepsilon, y_\varepsilon) \to R$ as $\varepsilon \to 0$, $R \in [0, +\infty)$. Then, thanks to (12.3.18), (Φ_ε) is bounded.

Case 2: we assume that $y_\varepsilon \to y_0$ as $\varepsilon \to 0$ where $y_0 \neq x_0$. We let $\delta > 0$ be such that $2\delta \le d_g(x_0, y_0)$, and write that

$$\int_M G_0(x, y_\varepsilon) u_\varepsilon^{2^\star - 1} dv_g \le \int_{B_{x_\varepsilon}(\delta)} H_\varepsilon u_\varepsilon^{2^\star - 1} dv_g + \int_{M\backslash B_{x_\varepsilon}(\delta)} H_\varepsilon u_\varepsilon^{2^\star - 1} dv_g$$

As above, standard properties of the Green function give that

$$\int_{B_{x_\varepsilon}(\delta)} H_\varepsilon u_\varepsilon^{2^\star - 1} dv_g \le C \int_{B_{x_\varepsilon}(\delta)} u_\varepsilon^{2^\star - 1} dv_g$$

and that

$$\int_{M\backslash B_{x_\varepsilon}(\delta)} H_\varepsilon u_\varepsilon^{2^\star - 1} dv_g \le C \int_{M\backslash B_{x_\varepsilon}(\delta)} d_g(x, y_\varepsilon)^{2-n} u_\varepsilon^{2^\star - 1} dv_g$$

where $C > 0$ is independent of ε. Thanks to (12.3.36),

$$\int_{M\backslash B_{x_\varepsilon}(\delta)} d_g(x, y_\varepsilon)^{2-n} u_\varepsilon^{2^\star - 1} dv_g = o\left(\mu_\varepsilon^{\frac{n}{2}-1}\right)$$

Independently, we can write that

$$\int_{B_{x_\varepsilon}(\delta)} u_\varepsilon^{2^\star-1} dv_g = \int_{B_{x_\varepsilon}(\mu_\varepsilon)} u_\varepsilon^{2^\star-1} dv_g + \int_{B_{x_\varepsilon}(\delta)\setminus B_{x_\varepsilon}(\mu_\varepsilon)} u_\varepsilon^{2^\star-1} dv_g$$

By (12.3.15),

$$\int_{B_{x_\varepsilon}(\mu_\varepsilon)} u_\varepsilon^{2^\star-1} dv_g = O\left(\mu_\varepsilon^{\frac{n}{2}-1}\right)$$

while by (12.3.36), taking $\nu > 0$ sufficiently small,

$$\int_{B_{x_\varepsilon}(\delta)\setminus B_{x_\varepsilon}(\mu_\varepsilon)} u_\varepsilon^{2^\star-1} dv_g = O\left(\mu_\varepsilon^{\frac{n}{2}-1}\right)$$

Coming back to (12.3.38), we get that (Φ_ε) is bounded.

Case 3: we assume that $\mu_\varepsilon^{-1} d_g(x_\varepsilon, y_\varepsilon) \to +\infty$ and that $d_g(x_\varepsilon, y_\varepsilon) \to 0$ as $\varepsilon \to 0$. We write that

$$\int_M G_0(x, y_\varepsilon) u_\varepsilon^{2^\star-1} dv_g \le \int_{\Omega_\varepsilon} H_\varepsilon u_\varepsilon^{2^\star-1} + \int_{M\setminus\Omega_\varepsilon} H_\varepsilon u_\varepsilon^{2^\star-1} dv_g$$

where $\Omega_\varepsilon = B_{y_\varepsilon}(\frac{d_g(x_\varepsilon,y_\varepsilon)}{2})$. As above, standard properties of the Green function and (12.3.36) give that

$$\begin{aligned}\int_M G_0(x, y_\varepsilon) u_\varepsilon^{2^\star-1} dv_g &\le C\mu_\varepsilon^{\frac{n+2}{2}(1-2\nu)} d_g(x_\varepsilon, y_\varepsilon)^{(n+2)(\nu-1)} \int_{\Omega_\varepsilon} d_g(x, y_\varepsilon)^{2-n} dv_g \\ &+ C d_g(x_\varepsilon, y_\varepsilon)^{2-n} \int_M u_\varepsilon^{2^\star-1} dv_g\end{aligned}$$

Then,

$$\begin{aligned}\int_M G_0(x, y_\varepsilon) u_\varepsilon^{2^\star-1} dv_g &\le C\mu_\varepsilon^{\frac{n+2}{2}(1-2\nu)} d_g(x_\varepsilon, y_\varepsilon)^{(n+2)(\nu-1)+2} \\ &+ C d_g(x_\varepsilon, y_\varepsilon)^{2-n} \mu_\varepsilon^{\frac{n}{2}-1}\end{aligned}$$

where $C > 0$ does not depend on ε. Coming back to (12.3.38), we get that

$$u_\varepsilon(y_\varepsilon)\mu_\varepsilon^{1-\frac{n}{2}} d_g(x_\varepsilon, y_\varepsilon)^{n-2} \le C\left(\frac{\mu_\varepsilon}{d_g(x_\varepsilon, y_\varepsilon)}\right)^{2-(n+2)\nu} + C$$

and since $\mu_\varepsilon^{-1} d_g(x_\varepsilon, y_\varepsilon) \to +\infty$ as $\varepsilon \to 0$, taking $\nu < \frac{2}{n+2}$, we get once again that (Φ_ε) is bounded.

Summarizing cases 1 to 3, we have proved that for any sequence (y_ε) in M, there exists $C > 0$, independent of ε, such that

$$u_\varepsilon(y_\varepsilon)\mu_\varepsilon^{1-\frac{n}{2}} d_g(x_\varepsilon, y_\varepsilon)^{n-2} \le C$$

This proves (12.3.37).

Thanks to (12.3.15), (12.3.16), and (12.3.37), integrating (12.3.5) over M and letting $\varepsilon \to 0$ give that

$$\lim_{\varepsilon\to 0} \hat{B}_\varepsilon \|u_\varepsilon\|_1 \|\Sigma_\varepsilon\|_1 \mu_\varepsilon^{1-\frac{n}{2}} = \frac{\omega_{n-1}}{n} A_n \tag{12.3.39}$$

where A_n is given by (12.1.36). Independently, let $\delta > 0$ small, and η be a smooth function such that $\eta = 0$ in $B_{x_0}(\frac{\delta}{2})$ and $\eta = 1$ in $M \backslash B_{x_0}(\delta)$. Multiplying (12.3.5) by η, and integrating over M, we get with (12.3.10) that

$$\hat{B}_\varepsilon \|u_\varepsilon\|_1 \int_M \eta \Sigma_\varepsilon dv_g = O\left(\|u_\varepsilon\|_1\right)$$

Since $\hat{B}_\varepsilon \to +\infty$ as $\varepsilon \to 0$, it follows that $\int_M \eta \Sigma_\varepsilon dv_g \to 0$ as $\varepsilon \to 0$. In particular,

$$\int_{M \backslash B_{x_0}(\delta)} \Sigma_\varepsilon dv_g \to 0$$

as $\varepsilon \to 0$. Since this holds for any $\delta > 0$ small, and since $0 \le \Sigma_\varepsilon \le 1$, we have proved that

$$\int_M \Sigma_\varepsilon dv_g \to 0 \tag{12.3.40}$$

as $\varepsilon \to 0$. We let $r_\varepsilon > 0$ be such that

$$\int_M \Sigma_\varepsilon dv_g = \frac{\omega_{n-1}}{n} r_\varepsilon^n \tag{12.3.41}$$

Then, $r_\varepsilon \to 0$ as $\varepsilon \to 0$, and thanks to (12.3.39),

$$\lim_{\varepsilon \to 0} \hat{B}_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^n \mu_\varepsilon^{1-\frac{n}{2}} = A_n \tag{12.3.42}$$

Now, for $x \in \mathcal{B}_0(\delta r_\varepsilon^{-1})$, $\delta > 0$ small, we define

$$\hat{g}_\varepsilon(x) = \left(\exp_{x_\varepsilon}^\star g\right)(r_\varepsilon x) \quad , \quad \hat{u}_\varepsilon(x) = r_\varepsilon^{\frac{n}{2}-1} u_\varepsilon\left(\exp_{x_\varepsilon}(r_\varepsilon x)\right)$$

and $\hat{\Sigma}_\varepsilon(x) = \Sigma_\varepsilon\left(\exp_{x_\varepsilon}(r_\varepsilon x)\right)$. Then,

$$\Delta_{\hat{g}_\varepsilon} \hat{u}_\varepsilon + \hat{B}_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \hat{\Sigma}_\varepsilon = \frac{1-\varepsilon}{K_n} \hat{u}_\varepsilon^{2^\star - 1} \tag{12.3.43}$$

Thanks to (12.3.41),

$$\limsup_{\varepsilon \to 0} \int_{\mathcal{B}_0(\delta r_\varepsilon^{-1})} \hat{\Sigma}_\varepsilon dv_{\hat{g}_\varepsilon} \le \frac{\omega_{n-1}}{n} \tag{12.3.44}$$

We set

$$\hat{\mu}_\varepsilon = \frac{\mu_\varepsilon}{r_\varepsilon} \tag{12.3.45}$$

It follows from (12.3.11) and (12.3.42) that $\hat{\mu}_\varepsilon \to 0$ as $\varepsilon \to 0$. Independently, (12.3.37) gives that

$$|x|^{n-2} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \hat{u}_\varepsilon(x) \le C \tag{12.3.46}$$

where $C > 0$ is independent of ε. By (12.3.43),

$$\Delta_{\hat{g}_\varepsilon}(\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \hat{u}_\varepsilon) + \hat{B}_\varepsilon \|u_\varepsilon\|_1 \hat{\mu}_\varepsilon^{1-\frac{n}{2}} r_\varepsilon^{\frac{n}{2}+1} \hat{\Sigma}_\varepsilon = \frac{1-\varepsilon}{K_n} \hat{\mu}_\varepsilon^2 (\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \hat{u}_\varepsilon)^{2^\star - 1} \tag{12.3.47}$$

and (12.3.42) gives that

$$\hat{B}_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \to A_n \tag{12.3.48}$$

as $\varepsilon \to 0$. Since $r_\varepsilon \to 0$ we also have that $\hat{g}_\varepsilon$ converges C^2 to the Euclidean metric in any compact subset of the Euclidean space. By (12.3.46), the $\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \hat{u}_\varepsilon$'s are bounded in any compact subset of $I\!R^n \backslash \{0\}$. By (12.3.47) and (12.3.48) they satisfy an equation with bounded coefficients. We can therefore assume that, up to a subsequence,

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \hat{u}_\varepsilon \to H \quad \text{in } C^1_{loc}\left(I\!R^n \backslash \{0\}\right) \tag{12.3.49}$$

where H is a solution of

$$\Delta H + A_n \hat{\Sigma} = 0 \quad \text{in } I\!R^n \backslash \{0\} \tag{12.3.50}$$

and $\hat{\Sigma}$ is such that $\hat{\Sigma}_\varepsilon \rightharpoonup \hat{\Sigma}$ in $L^p(I\!R^n)$ for any $p \geq 1$. Thanks to (12.3.44), and since $\hat{\Sigma}_\varepsilon \leq 1$, $\|\hat{\Sigma}\|_\infty \leq 1$. By (12.3.46) we clearly have that

$$|x|^{n-2} H(x) \leq C \tag{12.3.51}$$

for any $x \in I\!R^n \backslash \{0\}$, where $C > 0$ is independent of x.

We claim now that H can be computed explicitly. This is the subject of the following subsection.

12.3.3 An explicit expression for H

We claim first that H can be expressed as

$$H(x) = \frac{\lambda}{|x|^{n-2}} + \overline{H} \tag{12.3.52}$$

where λ is real and $\overline{H}$ is smooth. For the sake of completeness, we prove this elementary claim by using basic notions from the theory of harmonic functions. A possible reference for such notions is the excellent Han and Lin [23]. As a preliminary remark, we claim that a bounded harmonic function in $I\!R^n \backslash \mathcal{B}$, $n \geq 3$, has a limit at infinity. In order to prove this preliminary claim, one may proceed as follows. Let u be harmonic and bounded in $I\!R^n \backslash \mathcal{B}$. Up to replacing u by $u + A$, $A > 0$ a suitable constant, we can assume that u is nonnegative. Given $R > 1$, we let v_R be the smooth function in $\mathcal{B}_0(R)$ such that $\Delta v_R = 0$ in $\mathcal{B}_0(R)$ and $v_R = u$ on $\partial\mathcal{B}_0(R)$. When $|x| < R$, $v_R(x)$ is given by the Poisson integral formula

$$v_R(x) = \int_{\partial\mathcal{B}_0(R)} K(x,y) u(y) d\sigma(y)$$

The Poisson kernel K is such that $K \geq 0$ and

$$\int_{\partial\mathcal{B}_0(R)} K(x,y) d\sigma(y) = 1$$

for all $|x| < R$. In particular, we get that v_R is nonnegative and such that for any $x \in \mathcal{B}_0(R)$, $|v_R| \leq K$, where K is a bound for $|u|$ in $I\!R^n \backslash \mathcal{B}$. Given x and y two points in $I\!R^n$, and R large, the Harnack inequality for harmonic functions gives that

$$\lim_{R \to +\infty} \frac{v_R(y)}{v_R(x)} = 1 \tag{12.3.53}$$

We set now $w = u - v_R$, and let $r > 1$. Clearly, $|w(x)| \le K_r r^{n-2}|x|^{2-n}$ on $\partial\mathcal{B}_0(r)$, where K_r is the maximum of $|w|$ over $\partial\mathcal{B}_0(r)$. Since w and $\frac{1}{|x|^{n-2}}$ are harmonic in $\mathcal{B}_0(R)\backslash\mathcal{B}_0(r)$, the maximum principle gives that $|w(x)| \le K_r r^{n-2}|x|^{2-n}$ in $\mathcal{B}_0(R)\backslash\mathcal{B}_0(r)$. According to what we said above, $K_r \le 2K$. Hence,

$$|u(x) - v_R(x)| \le \frac{2Kr^{n-2}}{|x|^{n-2}} \tag{12.3.54}$$

in $\mathcal{B}_0(R)\backslash\mathcal{B}_0(r)$. We fix x in $I\!R^n$. Since the $v_R(x)$'s are bounded, there exists a sequence (R_k), with the property that $R_k \to +\infty$ as $k \to +\infty$, and there exists $\lambda \in I\!R$, such that $v_{R_k}(x) \to \lambda$ as $k \to +\infty$. Thanks to (12.3.53), we get that for any $x \in I\!R^n$, $v_{R_k}(x) \to \lambda$ as $k \to +\infty$. Coming back to (12.3.54), taking $R = R_k$, and passing to the limit $k \to +\infty$, we get that for any $x \in I\!R^n\backslash\mathcal{B}_0(r)$,

$$|u(x) - \lambda| \le \frac{2Kr^{n-2}}{|x|^{n-2}}$$

Then, $u(x) \to \lambda$ as $|x| \to +\infty$, and this proves our preliminary claim. We let now $u \in C^1(\mathcal{B})$ be such that $\Delta u = -A_n\hat{\Sigma}$ in $\mathcal{B}$, and set $\tilde{H} = H - u$. Then $\Delta\tilde{H} = 0$ in $\mathcal{B}\backslash\{0\}$. We let $\hat{H}$ be the Kelvin transform of $\tilde{H}$ given by

$$\hat{H}(x) = \frac{1}{|x|^{n-2}}\tilde{H}\left(\frac{x}{|x|^2}\right)$$

It is easily seen that $\Delta\hat{H} = 0$ in $I\!R^n\backslash\mathcal{B}$. Moreover, thanks to (12.3.51), $\hat{H}$ is bounded. The preliminary claim we just proved then gives that there exists λ real such that

$$\lim_{x\to 0}|x|^{n-2}\tilde{H}(x) = \lambda$$

Let Φ be given by

$$\Phi(x) = \tilde{H}(x) - \frac{\lambda}{|x|^{n-2}}$$

It is easily seen that Φ is harmonic in $\mathcal{B}\backslash\{0\}$, and, thanks to what we just proved, we have that $\Phi(x) = o\left(|x|^{2-n}\right)$. Standard arguments, see Han and Lin [23], then give that 0 is a removable singularity for Φ. This proves that H can be expressed as in (12.3.52), and thus our claim.

For convenience, we write that

$$H(x) = \frac{\lambda}{|x|^{n-2}} + \frac{A_n}{2n}|x|^2 + H_0(x) \tag{12.3.55}$$

where $H_0 \in C^1(I\!R^n)$ is such that

$$\Delta H_0 = A_n(1 - \hat{\Sigma}) \tag{12.3.56}$$

Let $r > 0$. By (12.3.15), (12.3.16) and (12.3.37),

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}_0(r)}\hat{u}_\varepsilon^{2^\star-1}dv_{\hat{g}_\varepsilon} \to \int_{I\!R^n}\tilde{u}^{2^\star-1}dx = \frac{\omega_{n-1}}{n}K_nA_n$$

as $\varepsilon \to 0$. Integrating (12.3.47) over $\mathcal{B}_0(r)$ and passing to the limit as $\varepsilon \to 0$ we then get that

$$-\int_{\partial\mathcal{B}_0(r)}\partial_\nu H d\sigma + A_n\int_{\mathcal{B}_0(r)}\hat{\Sigma}dx = \frac{\omega_{n-1}}{n}A_n \tag{12.3.57}$$

Since $\int_{\mathcal{B}_0(r)} \hat{\Sigma} dx \to 0$ as $r \to 0$, and

$$-\int_{\partial \mathcal{B}_0(r)} \partial_\nu H d\sigma \to (n-2)\lambda \omega_{n-1}$$

as $r \to 0$, it follows from (12.3.57) that

$$\lambda = \frac{A_n}{n(n-2)} \tag{12.3.58}$$

Noting that $H \geq 0$, we get with (12.3.58) that

$$H_0 \geq -\frac{A_n}{2(n-2)} \quad \text{on } \partial \mathcal{B}$$

By (12.3.56) and the maximum principle, since $\|\hat{\Sigma}\|_\infty \leq 1$, we get that

$$H_0 \geq -\frac{A_n}{2(n-2)} \quad \text{in } \mathcal{B}$$

In particular, $H(x) > 0$ for any $x \in \mathcal{B} \backslash \{0\}$, and it follows from (12.3.49) that for any r_1 and r_2 such that $0 < r_1 < r_2 < 1$, $\hat{u}_\varepsilon > 0$ in $\mathcal{B}_0(r_2) \backslash \mathcal{B}_0(r_1)$ for $\varepsilon > 0$ small. Then, $\hat{\Sigma}_\varepsilon = 1$ in $\mathcal{B}_0(r_2) \backslash \mathcal{B}_0(r_1)$ for $\varepsilon > 0$ small, and we get that

$$\int_{\mathcal{B}_0(r_2) \backslash \mathcal{B}_0(r_1)} \hat{\Sigma} dx = |\mathcal{B}_0(r_2) \backslash \mathcal{B}_0(r_1)|$$

where $|\mathcal{B}_0(r_2) \backslash \mathcal{B}_0(r_1)|$ is the Euclidean volume of $\mathcal{B}_0(r_2) \backslash \mathcal{B}_0(r_1)$. Letting $r_1 \to 0$ and $r_2 \to 1$, we then get that

$$\int_{\mathcal{B}} \hat{\Sigma} dx = \frac{\omega_{n-1}}{n}$$

We have $\|\hat{\Sigma}\|_\infty \leq 1$, and $\int_{\mathbb{R}^n} |\hat{\Sigma}| dx \leq n^{-1} \omega_{n-1}$. Thus,

$$\begin{aligned} &\hat{\Sigma} = 1 \ \text{ in } \mathcal{B} \\ &\hat{\Sigma} = 0 \ \text{ in } \mathbb{R}^n \backslash \mathcal{B} \end{aligned} \tag{12.3.59}$$

In particular, for any annulus $A \subset \mathbb{R}^n \backslash \mathcal{B}$, we get with (12.3.46) that

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_A \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} = \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_A \hat{\Sigma}_\varepsilon \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} \leq C \int_A \hat{\Sigma}_\varepsilon dv_{\hat{g}_\varepsilon}$$

and since $\int_A \hat{\Sigma}_\varepsilon dv_{\hat{g}_\varepsilon} \to \int_A \hat{\Sigma} dx$ as $\varepsilon \to 0$, we get with (12.3.59) that $H = 0$ in $\mathbb{R}^n \backslash \mathcal{B}$. Since H is C^1 in $\mathbb{R}^n \backslash \{0\}$, this implies that

$$\begin{aligned} &H(x) = \frac{A_n}{n(n-2)} \left(|x|^{2-n} - 1 \right) + \frac{A_n}{2n} \left(|x|^2 - 1 \right) \ \text{ in } \mathcal{B} \\ &H(x) = 0 \ \text{ in } \mathbb{R}^n \backslash \mathcal{B} \end{aligned} \tag{12.3.60}$$

and we have an explicit expression for H.

Thanks to (12.3.37),

$$\begin{aligned} r_\varepsilon^{-1-\frac{n}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{M\backslash B_{x_\varepsilon}(r_\varepsilon)} u_\varepsilon dv_g &= r_\varepsilon^{-1-\frac{n}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{M\backslash B_{x_\varepsilon}(r_\varepsilon)} \Sigma_\varepsilon u_\varepsilon dv_g \\ &\le Cr_\varepsilon^{-2}\int_{M\backslash B_{x_\varepsilon}(r_\varepsilon)} \frac{\Sigma_\varepsilon}{d_g(x_\varepsilon,x)^{n-2}}dv_g \\ &\le Cr_\varepsilon^{-n}\int_{M\backslash B_{x_\varepsilon}(r_\varepsilon)} \Sigma_\varepsilon dv_g \end{aligned}$$

and since

$$r_\varepsilon^{-n}\int_{B_{x_\varepsilon}(r_\varepsilon)}\Sigma_\varepsilon dv_g = \int_{\mathcal{B}}\hat{\Sigma}_\varepsilon dv_{\hat{g}_\varepsilon} \to \int_{\mathcal{B}}\hat{\Sigma}dx = \frac{\omega_{n-1}}{n}$$

as $\varepsilon \to 0$, we get with (12.3.41) that

$$r_\varepsilon^{-n}\int_{M\backslash B_{x_\varepsilon}(r_\varepsilon)}\Sigma_\varepsilon dv_g \to 0$$

as $\varepsilon \to 0$. Hence,

$$r_\varepsilon^{-1-\frac{n}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{M\backslash B_{x_\varepsilon}(r_\varepsilon)} u_\varepsilon dv_g \to 0 \tag{12.3.61}$$

as $\varepsilon \to 0$. Independently, given $\delta > 0$,

$$r_\varepsilon^{-1-\frac{n}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{B_{x_\varepsilon}(\delta r_\varepsilon)} u_\varepsilon dv_g = \hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}_0(\delta)}\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}$$

and it follows from (12.3.46) that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} r_\varepsilon^{-1-\frac{n}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{B_{x_\varepsilon}(\delta r_\varepsilon)} u_\varepsilon dv_g = 0 \tag{12.3.62}$$

At last,

$$r_\varepsilon^{-1-\frac{n}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{B_{x_\varepsilon}(r_\varepsilon)\backslash B_{x_\varepsilon}(\delta r_\varepsilon)} u_\varepsilon dv_g = \hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}\backslash\mathcal{B}_0(\delta)}\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}$$

and we get with (12.3.49) that for any $\delta \in (0,1)$,

$$r_\varepsilon^{-1-\frac{n}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{B_{x_\varepsilon}(r_\varepsilon)\backslash B_{x_\varepsilon}(\delta r_\varepsilon)} u_\varepsilon dv_g \to \int_{\mathcal{B}\backslash\mathcal{B}_0(\delta)} Hdx \tag{12.3.63}$$

as $\varepsilon \to 0$. Combining (12.3.61)-(12.3.63), letting $\delta \to 0$, we get with (12.3.60) that

$$r_\varepsilon^{-1-\frac{n}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\|u_\varepsilon\|_1 \to \int_{\mathcal{B}} Hdx = \frac{\omega_{n-1}}{2n(n+2)}A_n \tag{12.3.64}$$

as $\varepsilon \to 0$. Then, combining (12.3.48) and (12.3.64),

$$\hat{B}_\varepsilon r_\varepsilon^{n+2} \to \frac{2n(n+2)}{\omega_{n-1}} \tag{12.3.65}$$

as $\varepsilon \to 0$.

Going on with the asymptotic study of $\hat{u}_\varepsilon$, we prove sharp asymptotic estimates in the following subsection.

12.3.4 Strong Estimates.2

We claim that for any $\delta > 0$ there exists $C(\delta) > 1$ such that for $\varepsilon > 0$ small and any $x \in \mathcal{B}_0(\delta)$,

$$\frac{1}{C(\delta)}\left(\frac{\hat{\mu}_\varepsilon}{\hat{\mu}_\varepsilon^2 + \frac{\omega_n^{2/n}}{4}|x|^2}\right)^{\frac{n-2}{2}} \le \hat{u}_\varepsilon(x) \le C(\delta)\left(\frac{\hat{\mu}_\varepsilon}{\hat{\mu}_\varepsilon^2 + \frac{\omega_n^{2/n}}{4}|x|^2}\right)^{\frac{n-2}{2}} \tag{12.3.66}$$

with the property that $C(\delta) \to 1$ as $\delta \to 0$. Let us define U_ε by

$$U_\varepsilon(x) = \left(\frac{\hat{\mu}_\varepsilon}{\hat{\mu}_\varepsilon^2 + \frac{\omega_n^{2/n}}{4}|x|^2}\right)^{\frac{n-2}{2}}$$

and let (y_ε) be a sequence in $\mathcal{B}$. Suppose that $y_\varepsilon \to y_0$ as $\varepsilon \to 0$, $y_0 \neq 0$. Then, thanks to (12.3.49) and (12.3.60),

$$\frac{\hat{u}_\varepsilon(y_\varepsilon)}{U_\varepsilon(y_\varepsilon)} = 1 + 0(|y_\varepsilon|^{n-2})$$

In order to prove (12.3.66) it thus suffices to proves that if $y_\varepsilon \to 0$ as $\varepsilon \to 0$, then

$$\lim_{\varepsilon\to 0}\frac{\hat{u}_\varepsilon(y_\varepsilon)}{U_\varepsilon(y_\varepsilon)} = 1 \tag{12.3.67}$$

If $|y_\varepsilon| \le C\hat{\mu}_\varepsilon$, (12.3.67) follows from (12.3.15). In order to prove (12.3.67), and so (12.3.66), we are therefore left with the case where $y_\varepsilon \to 0$ and $\frac{|y_\varepsilon|}{\hat{\mu}_\varepsilon} \to +\infty$ as $\varepsilon \to 0$. Let $\hat{v}_\varepsilon$ be given by

$$\hat{v}_\varepsilon(x) = |y_\varepsilon|^{\frac{n}{2}-1}\hat{u}_\varepsilon(|y_\varepsilon|x)$$

and let

$$\hat{h}_\varepsilon(x) = \hat{g}_\varepsilon(|y_\varepsilon|x) \quad , \quad \sigma_\varepsilon(x) = \hat{\Sigma}_\varepsilon(|y_\varepsilon|x)$$

It is easily seen that

$$\Delta_{\hat{h}_\varepsilon}\hat{v}_\varepsilon + \hat{B}_\varepsilon(r_\varepsilon|y_\varepsilon|)^{\frac{n+2}{2}}\|u_\varepsilon\|_1\sigma_\varepsilon = \frac{1-\varepsilon}{K_n}\hat{v}_\varepsilon^{2^\star-1}$$

and that $\hat{h}_\varepsilon \to \xi$ in $C^1_{loc}(I\!R^n)$ as $\varepsilon \to 0$. We set

$$\hat{w}_\varepsilon(x) = \left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\right)^{1-\frac{n}{2}}\hat{v}_\varepsilon$$

Then,

$$\Delta_{\hat{h}_\varepsilon}\hat{w}_\varepsilon + |y_\varepsilon|^n\hat{B}_\varepsilon r_\varepsilon^{\frac{n+2}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\|u_\varepsilon\|_1\sigma_\varepsilon = \frac{1-\varepsilon}{K_n}\left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\right)^2\hat{w}_\varepsilon^{2^\star-1} \tag{12.3.68}$$

Thanks to (12.3.15) and (12.3.37),

$$\left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\right)^{n-2}\hat{w}_\varepsilon\left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}x\right) \to \tilde{u}(x) \tag{12.3.69}$$

in $C^0_{loc}(I\!R^n)$ as $\varepsilon \to 0$, and

$$|x|^{n-2}\hat{w}_\varepsilon(x) \le C \tag{12.3.70}$$

Thanks to (12.3.64) and (12.3.65), we also have that

$$|y_\varepsilon|^n \hat{B}_\varepsilon r_\varepsilon^{\frac{n+2}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \|u_\varepsilon\|_1 \to 0 \tag{12.3.71}$$

as $\varepsilon \to 0$. Noting that by (12.3.70), $\hat{w}_\varepsilon$ is bounded in any compact subset of $I\!R^n\backslash\{0\}$, it follows from standard elliptic theory, from (12.3.68), and (12.3.71), that $\hat{w}_\varepsilon \to \Psi$ in $C^1_{loc}(I\!R^n\backslash\{0\})$, where Ψ is a solution of $\Delta\Psi = 0$ in $I\!R^n\backslash\{0\}$. We let $\delta > 0$ small, and we integrate (12.3.68) over $\mathcal{B}_0(\delta)$. Then

$$\begin{aligned} &-\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \hat{w}_\varepsilon d\sigma_{\hat{h}_\varepsilon} + |y_\varepsilon|^n \hat{B}_\varepsilon r_\varepsilon^{\frac{n+2}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \|u_\varepsilon\|_1 \int_{\mathcal{B}_0(\delta)} \sigma_\varepsilon dv_{\hat{h}_\varepsilon} \\ &\quad = \frac{1-\varepsilon}{K_n}\left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\right)^2 \int_{\mathcal{B}_0(\delta)} \hat{w}_\varepsilon^{2^\star-1} dv_{\hat{h}_\varepsilon} \end{aligned} \tag{12.3.72}$$

It is easily seen that

$$\left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\right)^2 \int_{\mathcal{B}_0(\delta)} \hat{w}_\varepsilon^{2^\star-1} dv_{\hat{h}_\varepsilon} = \int_{\mathcal{B}_0(\delta\frac{|y_\varepsilon|}{\hat{\mu}_\varepsilon})} \left(\left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\right)^{n-2} \hat{w}_\varepsilon\left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}x\right)\right)^{2^\star-1} dv_{\tilde{g}_\varepsilon}$$

Thanks to (12.3.69) and (12.3.70) we then get that

$$\left(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\right)^2 \int_{\mathcal{B}_0(\delta)} \hat{w}_\varepsilon^{2^\star-1} dv_{\hat{h}_\varepsilon} \to \int_{I\!R^n} \tilde{u}^{2^\star-1} dx \tag{12.3.73}$$

as $\varepsilon \to 0$. Thanks to (12.3.71) we have that

$$|y_\varepsilon|^n \hat{B}_\varepsilon r_\varepsilon^{\frac{n+2}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \|u_\varepsilon\|_1 \int_{\mathcal{B}_0(\delta)} \sigma_\varepsilon dv_{\hat{h}_\varepsilon} \to 0 \tag{12.3.74}$$

as $\varepsilon \to 0$, and since $\hat{h}_\varepsilon \to \xi$ as $\varepsilon \to 0$, we also have that

$$\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \hat{w}_\varepsilon d\sigma_{\hat{h}_\varepsilon} \to \int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \Psi d\sigma_\xi \tag{12.3.75}$$

as $\varepsilon \to 0$. Combining (12.3.72)-(12.3.75), it follows that

$$\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \Psi d\sigma_\xi + \frac{1}{K_n}\int_{I\!R^n} \tilde{u}^{2^\star-1} dx = 0$$

and thus that

$$\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \Psi d\sigma_\xi + \frac{\omega_{n-1}}{n} A_n = 0 \tag{12.3.76}$$

As in subsection 12.1, (12.3.70) and (12.3.76) imply that

$$\Psi(x) = \frac{A_n}{n(n-2)|x|^{n-2}}$$

In particular, taking $x = y_\varepsilon/|y_\varepsilon|$, we get that for any sequence (y_ε) such that $y_\varepsilon \to 0$ and $\frac{|y_\varepsilon|}{\hat\mu_\varepsilon} \to +\infty$ as $\varepsilon \to 0$,

$$|y_\varepsilon|^{n-2}\hat\mu_\varepsilon^{1-\frac{n}{2}}\hat u_\varepsilon(y_\varepsilon) \to \frac{A_n}{n(n-2)} = 2^{n-2}\omega_n^{\frac{2}{n}-1} \tag{12.3.77}$$

as $\varepsilon \to 0$. This proves (12.3.67), and thus also (12.3.66).

From now on, we let $\mathcal{B}_2$ be the Euclidean ball $\mathcal{B}_0(2)$, and let $\eta \in C_c^\infty(\mathcal{B}_2)$ be a radially symmetrical function such that $\eta = 1$ in $\mathcal{B}$. We want to estimate

$$I = \int_{\mathcal{B}_2}(\eta\hat u_\varepsilon)^{2^\star}dx \quad\text{and}\quad J = \int_{\mathcal{B}_2}|\nabla(\eta\hat u_\varepsilon)|^2dx \tag{12.3.78}$$

We start with I.

12.3.5 An expansion for $\int_{\mathcal{B}_2}(\eta\hat u_\varepsilon)^{2^\star}dx$ as $\varepsilon \to 0$

We write that

$$\begin{aligned}\int_{\mathcal{B}_2}(\eta\hat u_\varepsilon)^{2^\star}dx &= \int_{\mathcal{B}}\hat u_\varepsilon^{2^\star}dx + \int_{\mathcal{B}_2\backslash\mathcal{B}}(\eta\hat u_\varepsilon)^{2^\star}dx\\ &= \int_{\mathcal{B}}\hat u_\varepsilon^{2^\star}(1-\sqrt{|\hat g_\varepsilon|})dx + \int_{\mathcal{B}}\hat u_\varepsilon^{2^\star}dv_{\hat g_\varepsilon} + \int_{\mathcal{B}_2\backslash\mathcal{B}}(\eta\hat u_\varepsilon)^{2^\star}dx\end{aligned}$$

where $|\hat g_\varepsilon|$ is the determinant of the components of $\hat g_\varepsilon$ in Euclidean coordinates. Thanks to the Cartan expansion of a metric in geodesic normal coordinates, we can write that

$$\sqrt{|\hat g_\varepsilon|} = 1 - \frac{r_\varepsilon^2}{6}R_{ij}(x_\varepsilon)x^ix^j + r_\varepsilon^3O(|x|^3)$$

where the R_{ij}'s are the components of the Ricci curvature of g in the exponential chart at x_ε. Then,

$$\begin{aligned}\int_{\mathcal{B}_2}(\eta\hat u_\varepsilon)^{2^\star}dx &= \int_{\mathcal{B}}\hat u_\varepsilon^{2^\star}dv_{\hat g_\varepsilon} + \frac{r_\varepsilon^2}{6}R_{ij}(x_\varepsilon)\int_{\mathcal{B}}\hat u_\varepsilon^{2^\star}x^ix^jdx\\ &+r_\varepsilon^3O\left(\int_{\mathcal{B}}|x|^3\hat u_\varepsilon^{2^\star}dx\right) + \int_{\mathcal{B}_2\backslash\mathcal{B}}(\eta\hat u_\varepsilon)^{2^\star}dx\end{aligned} \tag{12.3.79}$$

Thanks to (12.3.46),

$$\int_{\mathcal{B}_2\backslash\mathcal{B}}(\eta\hat u_\varepsilon)^{2^\star}dx = O(\hat\mu_\varepsilon^n) = o(\hat\mu_\varepsilon^{n-2}) \tag{12.3.80}$$

Similarly, it is easily seen that

$$\int_{\mathcal{B}}\hat u_\varepsilon^{2^\star}dv_{\hat g_\varepsilon} = 1 + o(\hat\mu_\varepsilon^{n-2}) \tag{12.3.81}$$

By (12.3.15),

$$\hat\mu_\varepsilon^{\frac{n}{2}-1}\hat u_\varepsilon(\hat\mu_\varepsilon x) \to \tilde u(x) \tag{12.3.82}$$

in $C^1_{loc}(I\!R^n)$, where $\tilde u$ is the fundamental solution given by (12.3.16). Combining this estimate with (12.3.46), it follows that

$$r_\varepsilon^3\int_{\mathcal{B}}|x|^3\hat u_\varepsilon^{2^\star}dx = o(r_\varepsilon^2\hat\mu_\varepsilon^2) \tag{12.3.83}$$

and that
$$\frac{r_\varepsilon^2}{6}R_{ij}(x_\varepsilon)\int_{\mathcal{B}}\hat{u}_\varepsilon^{2^\star}x^ix^jdx = \frac{S_g(x_0)}{6n}\left(\int_{I\!R^n}|x|^2\tilde{u}^{2^\star}dx\right)r_\varepsilon^2\hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2)$$
Noting that
$$\int_{I\!R^n}|x|^2\tilde{u}^{2^\star}dx = \frac{4n}{(n-2)\omega_n^{2/n}} = n^2K_n$$
we get that
$$\frac{r_\varepsilon^2}{6}R_{ij}(x_\varepsilon)\int_{\mathcal{B}}\hat{u}_\varepsilon^{2^\star}x^ix^jdx = \frac{nK_n}{6}S_g(x_0)r_\varepsilon^2\hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2) \tag{12.3.84}$$
Combining (12.3.79)-(12.3.81), (12.3.83), and (12.3.84), it follows that
$$I = 1 + \frac{nK_n}{6}S_g(x_0)r_\varepsilon^2\hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2) + o(\hat{\mu}_\varepsilon^{n-2}) \tag{12.3.85}$$
This is the expansion we were looking for.

We now compute an expansion for J. This is the subject of the following subsection.

12.3.6 An expansion for $\int_{\mathcal{B}_2}|\nabla(\eta\hat{u}_\varepsilon)|^2dx$ as $\varepsilon \to 0$

We write that
$$\begin{aligned}\int_{\mathcal{B}_2}|\nabla(\eta\hat{u}_\varepsilon)|_\xi^2dx &= \int_{\mathcal{B}_2}(\delta^{ij}-\hat{g}_\varepsilon^{ij})\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)dx\\ &+\int_{\mathcal{B}_2}|\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2(1-\sqrt{|\hat{g}_\varepsilon|})dx + \int_{\mathcal{B}_2}|\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2dv_{\hat{g}_\varepsilon}\end{aligned} \tag{12.3.86}$$
where the subscripts ξ and $\hat{g}_\varepsilon$ refer to the metric with respect to which the expression has to be understood. Thanks to (12.3.43), namely the equation satisfied by the $\hat{u}_\varepsilon$'s, we can write that
$$\begin{aligned}\int_{\mathcal{B}_2}|\nabla(\eta\hat{u}_\varepsilon)|^2dv_{\hat{g}_\varepsilon} &= \int_{\mathcal{B}_2}|\nabla\eta|^2\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon} + \int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon\Delta_{\hat{g}_\varepsilon}\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}\\ &= \frac{1-\varepsilon}{K_n}\int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon^{2^\star}dv_{\hat{g}_\varepsilon} - \hat{B}_\varepsilon\|u_\varepsilon\|_1r_\varepsilon^{\frac{n}{2}+1}\int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} + \int_{\mathcal{B}_2}|\nabla\eta|^2\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon}\end{aligned} \tag{12.3.87}$$
Thanks to (12.3.46) and (12.3.59),
$$\begin{aligned}\int_{\mathcal{B}_2}|\nabla\eta|^2\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon} &= O\left(\int_{\mathcal{B}_2\backslash\mathcal{B}}\hat{\Sigma}_\varepsilon\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon}\right)\\ &= \hat{\mu}_\varepsilon^{n-2}O\left(\int_{\mathcal{B}_2\backslash\mathcal{B}}\hat{\Sigma}_\varepsilon dv_{\hat{g}_\varepsilon}\right) = o\left(\hat{\mu}_\varepsilon^{n-2}\right)\end{aligned} \tag{12.3.88}$$
Independently, we can write that
$$\int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon^{2^\star}dv_{\hat{g}_\varepsilon} = \int_{\mathcal{B}}\hat{u}_\varepsilon^{2^\star}dv_{\hat{g}_\varepsilon} + \int_{\mathcal{B}_2\backslash\mathcal{B}}\eta^2\hat{u}_\varepsilon^{2^\star}dv_{\hat{g}_\varepsilon} = 1 + o(\hat{\mu}_\varepsilon^{n-2}) \tag{12.3.89}$$
At last, we write that
$$\begin{aligned}&\hat{B}_\varepsilon\|u_\varepsilon\|_1r_\varepsilon^{\frac{n}{2}+1}\int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}\\ &= \hat{\mu}_\varepsilon^{n-2}\left[r_\varepsilon^{-1-\frac{n}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\|u_\varepsilon\|_1\right]\left(\hat{B}_\varepsilon r_\varepsilon^{n+2}\right)\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}\end{aligned} \tag{12.3.90}$$

By (12.3.46),

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}_0(\delta)}\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}=0$$

With such a relation, it easily follows from (12.3.49) and (12.3.60) that

$$\lim_{\varepsilon\to 0}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}=\int_{\mathcal{B}}Hdx=\frac{\omega_{n-1}}{2n(n+2)}A_n \tag{12.3.91}$$

By (12.3.64), (12.3.65), and (12.3.91), coming back to (12.3.90), we then get that

$$\hat{B}_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}\int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}=\frac{\omega_{n-1}}{2n(n+2)}A_n^2\hat{\mu}_\varepsilon^{n-2}+o(\hat{\mu}_\varepsilon^{n-2}) \tag{12.3.92}$$

Finally, thanks to (12.3.88), (12.3.89), and (12.3.92), we get with (12.3.87) that

$$\int_{\mathcal{B}_2}|\nabla(\eta\hat{u}_\varepsilon)|^2 dv_{\hat{g}_\varepsilon}=\frac{1-\varepsilon}{K_n}-\frac{\omega_{n-1}}{2n(n+2)}A_n^2\hat{\mu}_\varepsilon^{n-2}+o(\hat{\mu}_\varepsilon^{n-2}) \tag{12.3.93}$$

Concerning the first term in the RHS of (12.3.86), we claim that

$$\int_{\mathcal{B}_2}(\delta^{ij}-\hat{g}_\varepsilon^{ij})\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)dx=o(r_\varepsilon^2\hat{\mu}_\varepsilon^2) \tag{12.3.94}$$

if $n\geq 5$, and that

$$\int_{\mathcal{B}_2}(\delta^{ij}-\hat{g}_\varepsilon^{ij})\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)dx=o(r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|) \tag{12.3.95}$$

if $n=4$. We assume first that $n\geq 5$. Thanks to the Cartan expansion of a metric in geodesic normal coordinates, we can write that

$$\hat{g}_\varepsilon^{ij}=\delta^{ij}-\frac{r_\varepsilon^2}{3}R^i{}_{\alpha\beta}{}^j(x_\varepsilon)x^\alpha x^\beta+r_\varepsilon^3 O(|x|^3)$$

where the R_{ijkl}'s are the components of the Riemann curvature tensor of g in the exponential chart at x_ε, and an index is raised with the metric. Let $\tilde{u}$ be given by (12.3.16). Since η and $\tilde{u}$ are radially symmetrical,

$$R^i{}_{\alpha\beta}{}^j(x_\varepsilon)\partial_i\left(\eta(\hat{\mu}_\varepsilon x)\tilde{u}(x)\right)\partial_j\left(\eta(\hat{\mu}_\varepsilon x)\tilde{u}(x)\right)x^\alpha x^\beta=0 \tag{12.3.96}$$

Let $R>0$. Thanks to (12.3.82), writing that $\int_{\mathcal{B}_2}=\int_{\mathcal{B}_0(R\hat{\mu}_\varepsilon)}+\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}$, it is easily seen with (12.3.96) that that for any $R>0$,

$$\begin{aligned}&\int_{\mathcal{B}_2}(\delta^{ij}-\hat{g}_\varepsilon^{ij})\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)dx\\&\quad\leq Cr_\varepsilon^2\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2|\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 dv_{\hat{g}_\varepsilon}+o(r_\varepsilon^2\hat{\mu}_\varepsilon^2)\end{aligned} \tag{12.3.97}$$

where $C>0$ does not depend on ε and R. We write that

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2|\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 dv_{\hat{g}_\varepsilon}\\&\quad\leq\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2|\nabla\eta|_{\hat{g}_\varepsilon}^2\hat{u}_\varepsilon^2 dv_{\hat{g}_\varepsilon}+\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\eta^2|\nabla\hat{u}_\varepsilon|_{\hat{g}_\varepsilon}^2 dv_{\hat{g}_\varepsilon}\end{aligned} \tag{12.3.98}$$

As in (12.3.88),
$$\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2|\nabla\eta|^2_{\hat{g}_\varepsilon}\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon}=o(\hat{\mu}_\varepsilon^{n-2}) \tag{12.3.99}$$
Independently, thanks to (12.3.82),
$$\int_{\partial\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\eta^2\hat{u}_\varepsilon|\partial_\nu\hat{u}_\varepsilon|d\sigma_{\hat{g}_\varepsilon}=\varepsilon_R\hat{\mu}_\varepsilon^2 \text{ and } \int_{\partial\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|\hat{u}_\varepsilon^2d\sigma_{\hat{g}_\varepsilon}=\varepsilon_R\hat{\mu}_\varepsilon^2$$
where ε_R is such that
$$\lim_{R\to+\infty}\lim_{\varepsilon\to0}\varepsilon_R=0$$
Hence,
$$\begin{aligned}\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\eta^2|\nabla\hat{u}_\varepsilon|^2_{\hat{g}_\varepsilon}dv_{\hat{g}_\varepsilon}&=\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\eta^2\hat{u}_\varepsilon\Delta_{\hat{g}_\varepsilon}\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}\\ &-\frac{1}{2}\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\Delta_{\hat{g}_\varepsilon}(|x|^2\eta^2)\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon}+\varepsilon_R\hat{\mu}_\varepsilon^2\end{aligned} \tag{12.3.100}$$
Thanks to (12.3.43), namely the equation satisfied by the $\hat{u}_\varepsilon$'s,
$$\begin{aligned}\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\eta^2\hat{u}_\varepsilon\Delta_{\hat{g}_\varepsilon}\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} &\le C\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\hat{u}_\varepsilon^{2^\star}dx\\ &= C\hat{\mu}_\varepsilon^2\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)}|x|^2\left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)\right)^{2^\star}dx\end{aligned}$$
so that, by (12.3.46),
$$\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\eta^2\hat{u}_\varepsilon\Delta_{\hat{g}_\varepsilon}\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}=\varepsilon_R\hat{\mu}_\varepsilon^2 \tag{12.3.101}$$
where ε_R is as above. Similarly,
$$\begin{aligned}\left|\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\Delta_{\hat{g}_\varepsilon}(|x|^2\eta^2)\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon}\right| &\le C\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\hat{u}_\varepsilon^2dx\\ &= C\hat{\mu}_\varepsilon^2\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)}\left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)\right)^2dx\end{aligned}$$
and still thanks to (12.3.46), since $n\ge5$, we get that
$$\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\Delta_{\hat{g}_\varepsilon}(|x|^2\eta^2)\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon}=\varepsilon_R\hat{\mu}_\varepsilon^2 \tag{12.3.102}$$
where ε_R is as above. Combining (12.3.97)-(12.3.102) we get that
$$\int_{\mathcal{B}_2}(\delta^{ij}-\hat{g}_\varepsilon^{ij})\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)dx=\varepsilon_Rr_\varepsilon^2\hat{\mu}_\varepsilon^2+o(r_\varepsilon^2\hat{\mu}_\varepsilon^2)$$
and since R is arbitrary, this proves (12.3.94). In order to prove (12.3.95), there we have $n=4$, we need to be more subtle. Still thanks to the Cartan expansion of a metric in geodesic normal

coordinates, we can write that

$$
\begin{aligned}
&\int_{\mathcal{B}_2}(\delta^{ij}-\hat{g}_\varepsilon^{ij})\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)dx\\
&\le Cr_\varepsilon^2\int_{\mathcal{B}_2}R^i{}_{\alpha\beta}{}^j(x_\varepsilon)\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)x^\alpha x^\beta dv_{\hat{g}_\varepsilon}\\
&+r_\varepsilon^2 o\left(\int_{\mathcal{B}_2}|x|^2|\nabla(\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}dv_{\hat{g}_\varepsilon}\right)
\end{aligned}
\tag{12.3.103}
$$

where $C>0$ does not depend on ε. Similar developments to the ones we made when $n\ge 5$ give that

$$
\int_{\mathcal{B}_2}|x|^2|\nabla(\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}dv_{\hat{g}_\varepsilon}=O\left(\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|\right)
\tag{12.3.104}
$$

when $n=4$. Independently, thanks to (12.3.96),

$$
\begin{aligned}
&\int_{\mathcal{B}_2}R^i{}_{\alpha\beta}{}^j(x_\varepsilon)\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)x^\alpha x^\beta dv_{\hat{g}_\varepsilon}\\
&\le\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}R^i{}_{\alpha\beta}{}^j(x_\varepsilon)\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)x^\alpha x^\beta dv_{\hat{g}_\varepsilon}+o(\hat{\mu}_\varepsilon^2)
\end{aligned}
\tag{12.3.105}
$$

where $R>0$ is fixed. Combining (12.3.103)-(12.3.105), we then get that for $R>0$,

$$
\begin{aligned}
&\int_{\mathcal{B}_2}(\delta^{ij}-\hat{g}_\varepsilon^{ij})\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)dx\\
&\le Cr_\varepsilon^2\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}R^i{}_{\alpha\beta}{}^j(x_\varepsilon)\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)x^\alpha x^\beta dv_{\hat{g}_\varepsilon}+o(r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|)
\end{aligned}
\tag{12.3.106}
$$

Let $K>0$ be an upper bound for the sectional curvature of g. Then,

$$
\begin{aligned}
&\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}R^i{}_{\alpha\beta}{}^j(x_\varepsilon)\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)x^\alpha x^\beta dv_{\hat{g}_\varepsilon}\\
&\le K\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\left(|\nabla(|x|\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}-(\nabla(|x|\eta\hat{u}_\varepsilon),\nu)^2_{\hat{g}_\varepsilon}\right)dv_{\hat{g}_\varepsilon}\\
&+o\left(\int_{\mathcal{B}_2}|x|^2|\nabla(\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}dv_{\hat{g}_\varepsilon}\right)
\end{aligned}
$$

where $\nu=\frac{x}{|x|}$, and thanks to (12.3.104), we get that

$$
\begin{aligned}
&\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}R^i{}_{\alpha\beta}{}^j(x_\varepsilon)\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)x^\alpha x^\beta dv_{\hat{g}_\varepsilon}\\
&\le K\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\left(|\nabla(|x|\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}-(\nabla(|x|\eta\hat{u}_\varepsilon),\nu)^2_{\hat{g}_\varepsilon}\right)dv_{\hat{g}_\varepsilon}+o(r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|)
\end{aligned}
\tag{12.3.107}
$$

It is easily seen that

$$
\begin{aligned}
&\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\left(|\nabla(|x|\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}-(\nabla(|x|\eta\hat{u}_\varepsilon),\nu)^2_{\hat{g}_\varepsilon}\right)dv_{\hat{g}_\varepsilon}\\
&=\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\eta^2\left(|\nabla(|x|\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}-(\nabla(|x|\hat{u}_\varepsilon),\nu)^2_{\hat{g}_\varepsilon}\right)dv_{\hat{g}_\varepsilon}+o(r_\varepsilon^2\hat{\mu}_\varepsilon^2)
\end{aligned}
\tag{12.3.108}
$$

Combining (12.3.106)-(12.3.108), it follows that

$$\begin{aligned}&\int_{\mathcal{B}_2}(\delta^{ij}-\hat{g}_\varepsilon^{ij})\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)dx\\&\quad\le Cr_\varepsilon^2\int_{\mathcal{B}_2\setminus\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\eta^2\left(|\nabla(|x|\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}-(\nabla(|x|\hat{u}_\varepsilon),\nu)^2_{\hat{g}_\varepsilon}\right)dv_{\hat{g}_\varepsilon}+o(r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|)\end{aligned}\tag{12.3.109}$$

Letting $\tilde{u}_\varepsilon$ be as in (12.3.12), $\tilde{u}_\varepsilon$ is given by

$$\tilde{u}_\varepsilon(x)=\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)$$

we have that

$$\begin{aligned}&\int_{\mathcal{B}_2\setminus\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\eta^2\left(|\nabla(|x|\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}-(\nabla(|x|\hat{u}_\varepsilon),\nu)^2_{\hat{g}_\varepsilon}\right)dv_{\hat{g}_\varepsilon}\\&\quad=\hat{\mu}_\varepsilon^2\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2\left(|\nabla(|x|\tilde{u}_\varepsilon)|^2_{\tilde{g}_\varepsilon}-(\nabla(|x|\tilde{u}_\varepsilon),\nu)^2_{\tilde{g}_\varepsilon}\right)dv_{\tilde{g}_\varepsilon}\end{aligned}\tag{12.3.110}$$

We write now that

$$\begin{aligned}&\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2\left(|\nabla(|x|\tilde{u}_\varepsilon)|^2_{\tilde{g}_\varepsilon}-(\nabla(|x|\tilde{u}_\varepsilon),\nu)^2_{\tilde{g}_\varepsilon}\right)dv_{\tilde{g}_\varepsilon}\\&\quad\le C\int_{\partial\mathcal{B}_0(R)}|\nabla(|x|^2\tilde{u}_\varepsilon^2)|_\xi d\sigma_\xi+\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2|x|\tilde{u}_\varepsilon\Delta_{\tilde{g}_\varepsilon}(|x|\tilde{u}_\varepsilon)dv_{\tilde{g}_\varepsilon}\\&\qquad-\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2(\nabla(|x|\tilde{u}_\varepsilon),\nu)^2_{\tilde{g}_\varepsilon}dv_{\tilde{g}_\varepsilon}+C\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\tilde{u}_\varepsilon^2dv_{\tilde{g}_\varepsilon}\end{aligned}$$

and since

$$\Delta_{\tilde{g}_\varepsilon}(|x|\tilde{u}_\varepsilon)=|x|\Delta_{\tilde{g}_\varepsilon}\tilde{u}_\varepsilon+\tilde{u}_\varepsilon\Delta_{\tilde{g}_\varepsilon}|x|-\frac{2}{|x|}(\nabla\tilde{u}_\varepsilon,x)_{\tilde{g}_\varepsilon}$$

we get that

$$\begin{aligned}&\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2\left(|\nabla(|x|\tilde{u}_\varepsilon)|^2_{\tilde{g}_\varepsilon}-(\nabla(|x|\tilde{u}_\varepsilon),\nu)^2_{\tilde{g}_\varepsilon}\right)dv_{\tilde{g}_\varepsilon}\\&\quad\le C\int_{\partial\mathcal{B}_0(R)}|\nabla(|x|^2\tilde{u}_\varepsilon^2)|_\xi d\sigma_\xi+\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2|x|^2\tilde{u}_\varepsilon\Delta_{\tilde{g}_\varepsilon}\tilde{u}_\varepsilon dv_{\tilde{g}_\varepsilon}\\&\qquad+\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2|x|\tilde{u}_\varepsilon^2\Delta_{\tilde{g}_\varepsilon}(|x|)dv_{\tilde{g}_\varepsilon}+C\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\tilde{u}_\varepsilon^2dv_{\tilde{g}_\varepsilon}\\&\qquad-2\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2\tilde{u}_\varepsilon(\nabla\tilde{u}_\varepsilon,x)_{\tilde{g}_\varepsilon}dv_{\tilde{g}_\varepsilon}\\&\qquad-\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2\left((\nabla\tilde{u}_\varepsilon,x)_{\tilde{g}_\varepsilon}+\tilde{u}_\varepsilon\right)^2dv_{\tilde{g}_\varepsilon}\end{aligned}$$

Noting that

$$|x|\Delta_{\tilde{g}_\varepsilon}(|x|)\le-(n-1)+C\mu_\varepsilon^2|x|^2$$

and since

$$\Delta_{\tilde{g}_\varepsilon}\tilde{u}_\varepsilon+\hat{B}_\varepsilon\mu_\varepsilon^{\frac{n+2}{2}}\|u_\varepsilon\|_1\tilde{\Sigma}_\varepsilon=\frac{1-\varepsilon}{K_n}\tilde{u}_\varepsilon^{2^\star-1}$$

it follows from the above computations that

$$\begin{aligned}
&\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(R)}\eta(\hat\mu_\varepsilon x)^2\left(|\nabla(|x|\tilde u_\varepsilon)|^2_{\tilde g_\varepsilon}-(\nabla(|x|\tilde u_\varepsilon),\nu)^2_{\tilde g_\varepsilon}\right)dv_{\tilde g_\varepsilon}\\
&\le C\int_{\partial\mathcal{B}_0(R)}|\nabla(|x|^2\tilde u_\varepsilon^2)|_\xi d\sigma_\xi + C\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(R)}|x|^2\tilde u_\varepsilon^{2^\star}dv_{\tilde g_\varepsilon}\\
&\quad +Cr_\varepsilon^2\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(R)}\tilde u_\varepsilon^2 dv_{\tilde g_\varepsilon}+C\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(\frac{1}{\hat\mu_\varepsilon})}\tilde u_\varepsilon^2dv_{\tilde g_\varepsilon}\\
&\quad -(n-4)\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(R)}\eta(\hat\mu_\varepsilon x)^2\tilde u_\varepsilon^2dv_{\tilde g_\varepsilon}\\
&\quad -\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(R)}\eta(\hat\mu_\varepsilon x)^2\left((\nabla\tilde u_\varepsilon,x)_{\tilde g_\varepsilon}+2\tilde u_\varepsilon\right)^2dv_{\tilde g_\varepsilon}
\end{aligned}$$

and hence that

$$\begin{aligned}
&\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(R)}\eta(\hat\mu_\varepsilon x)^2\left(|\nabla(|x|\tilde u_\varepsilon)|^2_{\tilde g_\varepsilon}-(\nabla(|x|\tilde u_\varepsilon),\nu)^2_{\tilde g_\varepsilon}\right)dv_{\tilde g_\varepsilon}\\
&\le C\int_{\partial\mathcal{B}_0(R)}|\nabla(|x|^2\tilde u_\varepsilon^2)|_\xi d\sigma_\xi + C\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(R)}|x|^2\tilde u_\varepsilon^{2^\star}dv_{\tilde g_\varepsilon}\\
&\quad +Cr_\varepsilon^2\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(R)}\tilde u_\varepsilon^2 dv_{\tilde g_\varepsilon}+C\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(\frac{1}{\hat\mu_\varepsilon})}\tilde u_\varepsilon^2dv_{\tilde g_\varepsilon}
\end{aligned}\tag{12.3.111}$$

By (12.3.82),

$$\int_{\partial\mathcal{B}_0(R)}|\nabla(|x|^2\tilde u_\varepsilon^2)|_\xi d\sigma_\xi\le C\tag{12.3.112}$$

while by (12.3.43),

$$\begin{aligned}
&\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(R)}|x|^2\tilde u_\varepsilon^{2^\star}dv_{\tilde g_\varepsilon}\le C\\
&\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(R)}\tilde u_\varepsilon^2dv_{\tilde g_\varepsilon}=O(|\ln\hat\mu_\varepsilon|)\\
&\int_{\mathcal{B}_0(\frac{2}{\hat\mu_\varepsilon})\backslash\mathcal{B}_0(\frac{1}{\hat\mu_\varepsilon})}\tilde u_\varepsilon^2dv_{\tilde g_\varepsilon}\le C
\end{aligned}\tag{12.3.113}$$

when $n=4$. Combining (12.3.109)-(12.3.113), it follows that when $n=4$,

$$\int_{\mathcal{B}_2}(\delta^{ij}-\hat g_\varepsilon^{ij})\partial_i(\eta\hat u_\varepsilon)\partial_j(\eta\hat u_\varepsilon)dx=o(r_\varepsilon^2\hat\mu_\varepsilon^2|\ln\hat\mu_\varepsilon|)$$

and this proves (12.3.95). Still when estimating J, we now have to deal with the second term in the RHS of (12.3.86). We claim here that

$$\int_{\mathcal{B}_2}|\nabla(\eta\hat u_\varepsilon)|^2_{\hat g_\varepsilon}(1-\sqrt{|\hat g_\varepsilon|})dx=\frac{(n-2)(n+2)}{6(n-4)}S_g(x_0)r_\varepsilon^2\hat\mu_\varepsilon^2+o(r_\varepsilon^2\hat\mu_\varepsilon^2)\tag{12.3.114}$$

when $n\ge5$, and that

$$\int_{\mathcal{B}_2}|\nabla(\eta\hat u_\varepsilon)|^2_{\hat g_\varepsilon}(1-\sqrt{|\hat g_\varepsilon|})dx=\frac{8\omega_3}{3\omega_4}S_g(x_0)r_\varepsilon^2\hat\mu_\varepsilon^2|\ln\hat\mu_\varepsilon|+o(r_\varepsilon^2\hat\mu_\varepsilon^2|\ln\hat\mu_\varepsilon|)\tag{12.3.115}$$

when $n = 4$. In order to prove this claim, we write that

$$\sqrt{|\hat{g}_\varepsilon|} = 1 - \frac{r_\varepsilon^2}{6} R_{ij}(x_\varepsilon) x^i x^j + r_\varepsilon^3 O(|x|^3)$$

where the R_{ij}'s are the components of the Ricci curvature of g in the exponential chart at x_ε. Then,

$$\begin{aligned} &\int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon} (1 - \sqrt{|\hat{g}_\varepsilon|}) dx \\ &= \frac{r_\varepsilon^2}{6} R_{ij}(x_\varepsilon) \int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon} x^i x^j dv_{\hat{g}_\varepsilon} + r_\varepsilon^3 O\left(\int_{\mathcal{B}_2} |x|^2 |\nabla(\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon} dv_{\hat{g}_\varepsilon}\right) \end{aligned} \tag{12.3.116}$$

As above,

$$\int_{\mathcal{B}_2} |x|^2 |\nabla(\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon} dv_{\hat{g}_\varepsilon} = O(\hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|) \tag{12.3.117}$$

when $n = 4$, and

$$\int_{\mathcal{B}_2} |x|^2 |\nabla(\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon} dv_{\hat{g}_\varepsilon} = O(\hat{\mu}_\varepsilon^2) \tag{12.3.118}$$

when $n \geq 5$. Similarly, it is easily seen that

$$\begin{aligned} &R_{ij}(x_\varepsilon) \int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon} x^i x^j dv_{\hat{g}_\varepsilon} \\ &= R_{ij}(x_\varepsilon) \int_{\mathcal{B}_2} \eta^2 |\nabla\hat{u}_\varepsilon|^2_{\hat{g}_\varepsilon} x^i x^j dv_{\hat{g}_\varepsilon} + o(\hat{\mu}_\varepsilon^2) \end{aligned} \tag{12.3.119}$$

Then,

$$\begin{aligned} &R_{ij}(x_\varepsilon) \int_{\mathcal{B}_2} \eta^2 |\nabla\hat{u}_\varepsilon|^2_{\hat{g}_\varepsilon} x^i x^j dv_{\hat{g}_\varepsilon} \\ &= R_{ij}(x_\varepsilon) \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon \Delta_{\hat{g}_\varepsilon} \hat{u}_\varepsilon x^i x^j dv_{\hat{g}_\varepsilon} - \frac{1}{2} \int_{\mathcal{B}_2} \Delta_{\hat{g}_\varepsilon} (\eta^2 R_{ij}(x_\varepsilon) x^i x^j) \hat{u}_\varepsilon^2 dv_{\hat{g}_\varepsilon} \end{aligned} \tag{12.3.120}$$

By (12.3.43),

$$\begin{aligned} &R_{ij}(x_\varepsilon) \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon \Delta_{\hat{g}_\varepsilon} \hat{u}_\varepsilon x^i x^j dv_{\hat{g}_\varepsilon} \\ &= \frac{1-\varepsilon}{K_n} R_{ij}(x_\varepsilon) \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^{2^\star} x^i x^j dv_{\hat{g}_\varepsilon} \\ &- \hat{B}_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} R_{ij}(x_\varepsilon) \int_{\mathcal{B}_2} \eta^2 \hat{\Sigma}_\varepsilon \hat{u}_\varepsilon x^i x^j dv_{\hat{g}_\varepsilon} \end{aligned}$$

As when getting (12.3.92),

$$\hat{B}_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} R_{ij}(x_\varepsilon) \int_{\mathcal{B}_2} \eta^2 \hat{\Sigma}_\varepsilon \hat{u}_\varepsilon x^i x^j dv_{\hat{g}_\varepsilon} = o(\hat{\mu}_\varepsilon^2)$$

while, thanks to (12.3.46) and (12.3.82),

$$R_{ij}(x_\varepsilon) \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^{2^\star} x^i x^j dv_{\hat{g}_\varepsilon} = \frac{S_g(x_0)}{n} \int_{\mathbb{R}^n} |x|^2 \tilde{u}^{2^\star} dx + o(\hat{\mu}_\varepsilon^2)$$

Noting that

$$\int_{\mathbb{R}^n} |x|^2 \tilde{u}^{2^\star} dx = n^2 K_n$$

it follows that

$$R_{ij}(x_\varepsilon) \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon \Delta_{\hat{g}_\varepsilon} \hat{u}_\varepsilon x^i x^j dv_{\hat{g}_\varepsilon} = nK_n S_g(x_0)\hat{\mu}_\varepsilon^2 + o(\hat{\mu}_\varepsilon^2) \tag{12.3.121}$$

Independently,

$$\int_{\mathcal{B}_2} \Delta_{\hat{g}_\varepsilon}(\eta^2 R_{ij}(x_\varepsilon)x^i x^j)\hat{u}_\varepsilon^2 dv_{\hat{g}_\varepsilon} = -2S_g(x_0)\left(1+o(1)\right)\int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx + o(\hat{\mu}_\varepsilon^2)$$

When $n \geq 5$, we get with (12.3.46) and (12.3.82) that

$$\int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx = \hat{\mu}_\varepsilon^2 \int_{\mathbb{R}^n} \tilde{u}^2 dx + o(\hat{\mu}_\varepsilon^2)$$

Since, see (12.1.41),

$$\int_{\mathbb{R}^n} \tilde{u}^2 dx = \frac{4(n-1)}{n-4}$$

it follows that when $n \geq 5$,

$$\int_{\mathcal{B}_2} \Delta_{\hat{g}_\varepsilon}(\eta^2 R_{ij}(x_\varepsilon)x^i x^j)\hat{u}_\varepsilon^2 dv_{\hat{g}_\varepsilon} = -\frac{8(n-1)}{n-4} S_g(x_0)\hat{\mu}_\varepsilon^2 + o(\hat{\mu}_\varepsilon^2) \tag{12.3.122}$$

Combining (12.3.116) and (12.3.118)-(12.3.122), we get that when $n \geq 5$,

$$\int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 (1 - \sqrt{|\hat{g}_\varepsilon|})dx = \frac{n^2-4}{6(n-4)} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2)$$

and this proves (12.3.114). When $n = 4$, we use (12.3.66). As in (12.1.55), it follows from (12.3.66) that

$$\int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx = \frac{16\omega_3}{\omega_4} \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o\left(\hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|\right) \tag{12.3.123}$$

Combining (12.3.116), (12.3.117), (12.3.119)-(12.3.121), and (12.3.123), we get that when $n = 4$,

$$\int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 (1 - \sqrt{|\hat{g}_\varepsilon|})dx = \frac{8\omega_3}{3\omega_4} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln \hat{\mu}_\varepsilon|)$$

This proves (12.3.115). Summarizing, it follows from (12.3.86), (12.3.93)-(12.3.95), (12.3.114), and (12.3.115) that

$$\begin{aligned} J = &\frac{1-\varepsilon}{K_n} - \frac{\omega_{n-1}}{2n(n+2)} A_n^2 \hat{\mu}_\varepsilon^{n-2} \\ &+ \frac{n^2-4}{6(n-4)} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2) + o(\hat{\mu}_\varepsilon^{n-2}) \end{aligned} \tag{12.3.124}$$

when $n \geq 5$, and that

$$\begin{aligned} J = &\frac{1-\varepsilon}{K_n} - \frac{\omega_{n-1}}{2n(n+2)} A_n^2 \hat{\mu}_\varepsilon^{n-2} \\ &+ \frac{8\omega_3}{3\omega_4} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln \hat{\mu}_\varepsilon|) + o(\hat{\mu}_\varepsilon^{n-2}) \end{aligned} \tag{12.3.125}$$

when $n = 4$.

Now the proof of (12.3.4) and (12.3.5) proceeds as follows. We write that

$$\left(\int_{\mathcal{B}_2} (\eta \hat{u}_\varepsilon)^{2^\star} dx\right)^{\frac{2}{2^\star}} \leq K_n \int_{\mathcal{B}_2} |\nabla(\eta \hat{u}_\varepsilon)|^2 dx$$

namely that $I^{2/2^\star} \leq K_n J$. Thanks to (12.3.85), (12.3.124), and (12.3.125), we then get that

$$\begin{aligned} &\varepsilon + \frac{\omega_{n-1}}{2n(n+2)} A_n^2 K_n \hat{\mu}_\varepsilon^{n-2} \\ &\quad \leq \frac{n-2}{n-4} K_n S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2) + o(\hat{\mu}_\varepsilon^{n-2}) \end{aligned} \tag{12.3.126}$$

when $n \geq 5$, and

$$\begin{aligned} &\varepsilon + \frac{\omega_3}{48} K_4^2 A_4^2 \hat{\mu}_\varepsilon^2 \\ &\quad \leq \frac{8\omega_3}{3\omega_4} K_4^2 S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|) + o(\hat{\mu}_\varepsilon^2) \end{aligned} \tag{12.3.127}$$

when $n = 4$. A direct consequence of (12.3.126) and (12.3.127) is that $S_g(x_0) \geq 0$. We claim that $S_g(x_0) > 0$. Let us assume first that $n \geq 5$. Writing that

$$\left(\hat{B}_\varepsilon r_\varepsilon^{n+2}\right)^{\frac{2}{n+2}} = \left(\varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} \hat{B}_\varepsilon\right)^{\frac{2}{n+2}} \varepsilon^{\frac{4-n}{n-2}} r_\varepsilon^2$$

it follows from (12.3.65) and (12.2.3) that

$$\limsup_{\varepsilon \to 0} \varepsilon^{\frac{4-n}{n-2}} r_\varepsilon^2 < +\infty \tag{12.3.128}$$

Thanks to (12.3.126), assuming that $S_g(x_0) = 0$, we then get with (12.3.128) that

$$\varepsilon \hat{\mu}_\varepsilon^{2-n} = o\left(r_\varepsilon^2 \hat{\mu}_\varepsilon^{4-n}\right) = o\left((\varepsilon \hat{\mu}_\varepsilon^{2-n})^{\frac{n-4}{n-2}}\right)$$

Hence, $\varepsilon \hat{\mu}_\varepsilon^{2-n} \to 0$ as $\varepsilon \to 0$, so that

$$r_\varepsilon^2 \hat{\mu}_\varepsilon^{4-n} = o\left(r_\varepsilon^2 \varepsilon^{\frac{4-n}{n-2}}\right)$$

and $r_\varepsilon^2 \hat{\mu}_\varepsilon^{4-n} \to 0$ as $\varepsilon \to 0$ thanks to (12.3.128). Coming back to (12.3.126), we get a contradiction. This proves the claim that $S_g(x_0) > 0$ in the case $n \geq 5$. When $n = 4$, (12.3.65) and (12.2.2) give that

$$\limsup_{\varepsilon \to 0} r_\varepsilon^2 |\ln \varepsilon| < +\infty \tag{12.3.129}$$

Combining (12.3.127) and (12.3.129), we can write that

$$\varepsilon |\ln \varepsilon| \leq C \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|$$

and this implies that

$$\frac{1}{|\ln \varepsilon|} = O\left(\frac{1}{|\ln \hat{\mu}_\varepsilon|}\right) \tag{12.3.130}$$

Coming back to (12.3.127), assuming that $S_g(x_0) = 0$, we get that

$$\frac{\omega_3}{48} K_4^2 A_4^2 + o(1) \le o\left(\frac{|\ln \hat{\mu}_\varepsilon|}{|\ln \varepsilon|}\right)$$

a contradiction thanks to (12.3.130). This proves the claim that $S_g(x_0) > 0$ in the case $n = 4$. Then it follows from (12.3.126) and (12.3.127) that

$$\liminf_{\varepsilon \to 0} r_\varepsilon^2 \hat{\mu}_\varepsilon^{4-n} > 0 \tag{12.3.131}$$

when $n \ge 5$, and

$$\liminf_{\varepsilon \to 0} r_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| > 0 \tag{12.3.132}$$

when $n = 4$. In particular, $\hat{\mu}_\varepsilon^{n-2} = O(r_\varepsilon^2 \hat{\mu}_\varepsilon^2)$ when $n \ge 5$, and $\hat{\mu}_\varepsilon^2 = O(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|)$ when $n = 4$. Coming back to (12.3.126) and (12.3.127) we then get that

$$\varepsilon = O(r_\varepsilon^2 \hat{\mu}_\varepsilon^2) \text{ when } n \ge 5 \text{ and } \varepsilon = O(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|) \text{ when } n = 4 \tag{12.3.133}$$

We now consider the sharp inequality of subsection 12.1. We choose α to be given by the equation $\alpha = \frac{n-2}{4(n-1)} S_g(x_0)$, and apply this inequality to the function

$$\varphi_\varepsilon(x) = \eta(\frac{x}{r_\varepsilon}) u_\varepsilon \left(\exp_{x_\varepsilon}(x)\right)$$

where η is as above. The change of variable $x = r_\varepsilon y$ then gives that

$$\begin{aligned} &\frac{1-\varepsilon}{K_n} \left(\int_{\mathcal{B}_2} (\eta \hat{u}_\varepsilon)^{2^\star} dx\right)^{\frac{2}{2^\star}} \\ &\le \int_{\mathcal{B}_2} |\nabla(\eta \hat{u}_\varepsilon)|^2 dx - \frac{n-2}{4(n-1)} S_g(x_0) r_\varepsilon^2 \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx \\ &\quad + B_\varepsilon r_\varepsilon^{n+2} \left(\int_{\mathcal{B}_2} \eta \hat{u}_\varepsilon dx\right)^2 \end{aligned} \tag{12.3.134}$$

Thanks to (12.3.85)-(12.3.89), (12.3.94), (12.3.95), (12.3.114), and (12.3.115), it follows from (12.3.134) that

$$\begin{aligned} &\hat{B}_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} - B_\varepsilon \left(\int_{\mathcal{B}_2} \eta \hat{u}_\varepsilon dx\right)^2 r_\varepsilon^{n+2} \\ &\le \frac{n-2}{n-4} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 - \frac{n-2}{4(n-1)} S_g(x_0) r_\varepsilon^2 \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx \\ &\quad + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2) + o(\hat{\mu}_\varepsilon^{n-2}) \end{aligned} \tag{12.3.135}$$

when $n \ge 5$, and

$$\begin{aligned} &\hat{B}_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} - B_\varepsilon \left(\int_{\mathcal{B}_2} \eta \hat{u}_\varepsilon dx\right)^2 r_\varepsilon^{n+2} \\ &\le \frac{8\omega_3}{3\omega_4} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| - \frac{n-2}{4(n-1)} S_g(x_0) r_\varepsilon^2 \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx \\ &\quad + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|) + o(\hat{\mu}_\varepsilon^{n-2}) \end{aligned} \tag{12.3.136}$$

when $n = 4$. We have already seen, see (12.3.122) and (12.3.123), that

$$\int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx = \frac{4(n-1)}{n-4} \hat{\mu}_\varepsilon^2 + o(\hat{\mu}_\varepsilon^2)$$

when $n \geq 5$, and

$$\int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx = \frac{16\omega_3}{\omega_4} \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o(\hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|)$$

when $n = 4$. Hence,

$$\frac{n-2}{4(n-1)} S_g(x_0) r_\varepsilon^2 \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx = \frac{n-2}{n-4} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2) \tag{12.3.137}$$

when $n \geq 5$, and

$$\begin{aligned} &\frac{n-2}{4(n-1)} S_g(x_0) r_\varepsilon^2 \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx \\ &= \frac{8\omega_3}{3\omega_4} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|) \end{aligned} \tag{12.3.138}$$

when $n = 4$. Independently, similar computations to the ones we made to get (12.3.64) give that

$$r_\varepsilon^{\frac{n}{2}+1} \|u_\varepsilon\|_1 \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} = \left(\int_{\mathcal{B}} H dx \right)^2 r_\varepsilon^{n+2} \hat{\mu}_\varepsilon^{n-2} + o(r_\varepsilon^{n+2} \hat{\mu}_\varepsilon^{n-2}) \tag{12.3.139}$$

and that

$$\int_{\mathcal{B}_2} \eta \hat{u}_\varepsilon dx = \left(\int_{\mathcal{B}} H dx \right) \hat{\mu}_\varepsilon^{\frac{n}{2}-1} + o(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}) \tag{12.3.140}$$

We have already seen that $\int_{\mathcal{B}} H dx > 0$. We also have that $\hat{\mu}_\varepsilon^{n-2} = O(r_\varepsilon^2 \hat{\mu}_\varepsilon^2)$ when $n \geq 5$, and $\hat{\mu}_\varepsilon^2 = O(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|)$ when $n = 4$. Combining (12.3.135)-(12.3.140) we then get that

$$\hat{B}_\varepsilon - B_\varepsilon + o(B_\varepsilon) \leq o(r_\varepsilon^{-n} \hat{\mu}_\varepsilon^{4-n}) \tag{12.3.141}$$

when $n \geq 5$, and

$$\hat{B}_\varepsilon - B_\varepsilon + o(B_\varepsilon) \leq o(r_\varepsilon^{-4} |\ln \hat{\mu}_\varepsilon|) \tag{12.3.142}$$

when $n = 4$. It easily follows from (12.3.131) and (12.3.133) that

$$r_\varepsilon^{-n} \hat{\mu}_\varepsilon^{4-n} \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} = O(1) \tag{12.3.143}$$

when $n \geq 5$, and it easily follows from (12.3.132) and (12.3.133) that

$$r_\varepsilon^{-4} |\ln \hat{\mu}_\varepsilon| |\ln \varepsilon|^{-3} = O(1) \tag{12.3.144}$$

Combining (12.3.140)-(12.3.144), we then get with (12.1.4) and (12.1.5) that

$$\limsup_{\varepsilon \to 0} \hat{B}_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} \leq C_n S_g(x_0)^{\frac{n+2}{2}} \tag{12.3.145}$$

when $n \geq 5$, and

$$\limsup_{\varepsilon \to 0} \frac{\hat{B}_\varepsilon}{|\ln \varepsilon|^3} \leq \frac{1}{2304\omega_3} S_g(x_0)^3 \tag{12.3.146}$$

when $n = 4$, where

$$C_n = \frac{2n(n+2)\omega_n^{2+\frac{4}{n}}}{\omega_{n-1}^{\frac{2n}{n-2}} \left(4^{n-3}n(n-2)(n-4)\right)^{\frac{n+2}{n-2}}}$$

Thanks to the results of subsection 12.2, namely (12.2.2) and (12.2.3), it follows from (12.3.145) and (12.3.146) that

$$S_g(x_0) = \max_{x \in M} S_g(x) \tag{12.3.147}$$

Combining (12.3.145)-(12.3.147) we then get that (12.3.3) and (12.3.4) are proved.

It is easily seen that the second part of Theorem 4.4 follows from the results of subsections 12.2 and 12.3. Combining (12.2.2)-(12.2.3) and (12.3.3)-(12.3.4), we indeed do get that

$$\lim_{\varepsilon \to 0} \frac{\hat{B}_\varepsilon}{|\ln \varepsilon|^3} = \frac{1}{2304\omega_3} \left(\max_{x \in M} S_g\right)^3$$

when $n = 4$, and

$$\lim_{\varepsilon \to 0} \hat{B}_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} = C_n \left(\max_{x \in M} S_g\right)^{\frac{n+2}{2}}$$

when $n \geq 5$. Thanks to (12.3.2), this ends the proof of the second part of Theorem 4.4.

Appendix

We prove Theorem 4.3 in this appendix, following Druet [16], private communication. We let (M, g) be a smooth compact Riemannian manifold of dimension $n = 4$ or $n = 5$, and of nonpositive scalar curvature. We let also $\hat{B}_\varepsilon$ be the smallest B such that for any $u \in H_1^2(M)$,

$$\frac{1-\varepsilon}{K_n} \|u\|_{2^\star}^2 \leq \|\nabla u\|_2^2 + B\|u\|_1^2$$

In order to prove Theorem 4.3, it suffices to prove that $\hat{B}_\varepsilon$ is bounded as $\varepsilon \to 0$. We proceed here by contradiction, and assume that $\hat{B}_\varepsilon \to +\infty$ as $\varepsilon \to 0$. The analysis of the preceding section can then be applied. In particular, the following holds. For any $\varepsilon > 0$, there exists $u_\varepsilon \in C^{1,\beta}(M)$, $0 < \beta < 1$, $u_\varepsilon \geq 0$, such that

$$\Delta_g u_\varepsilon + \hat{B}_\varepsilon \|u_\varepsilon\|_1 \Sigma_\varepsilon = \frac{1-\varepsilon}{K_n} u_\varepsilon^{2^\star - 1} \tag{A1}$$

and

$$\int_M u_\varepsilon^{2^\star} dv_g = 1 \tag{A2}$$

where $\Delta_g = -div_g(\nabla)$ is the Riemannian Laplacian, and $\Sigma_\varepsilon \in L^\infty(M)$, $0 \leq \Sigma_\varepsilon \leq 1$, is such that $\Sigma_\varepsilon u_\varepsilon = u_\varepsilon$. We let x_ε be a point where u_ε is maximum, and set

$$\mu_\varepsilon^{1-\frac{n}{2}} = \|u_\varepsilon\|_\infty = u_\varepsilon(x_\varepsilon) \tag{A3}$$

Then $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$, and $u_\varepsilon \to 0$ in $C^0_{loc}(M\backslash\{x_0\})$ as $\varepsilon \to 0$, where x_0 is the limit of the x_ε's as $\varepsilon \to 0$. Moreover,

$$\lim_{\varepsilon\to 0} \mu_\varepsilon^{\frac{n}{2}-1} u_\varepsilon \left(\exp_{x_\varepsilon}(\mu_\varepsilon x)\right) = V_0(|x|) \tag{A4}$$

in $C^1_{loc}(I\!R^n) \cap D^2_1(I\!R^n)$, where $V_0 : I\!R \to I\!R$ is given by

$$V_0(X) = \left(1 + \frac{\omega_n^{2/n}}{4} X^2\right)^{1-\frac{n}{2}}$$

We also have that the following sharp C^0-estimate holds: there exists $C > 0$ such that for any $\varepsilon > 0$ and any x,

$$\mu_\varepsilon^{1-\frac{n}{2}} d_g(x_\varepsilon, x)^{n-2} u_\varepsilon(x) \le C \tag{A5}$$

We let r_ε be such that

$$\int_M \Sigma_\varepsilon dv_g = \frac{\omega_{n-1}}{n} r_\varepsilon^n \tag{A6}$$

Then $r_\varepsilon \to 0$ as $\varepsilon \to 0$, and $\mu_\varepsilon r_\varepsilon^{-1} \to 0$ as $\varepsilon \to 0$.

From now on, we let $\eta : [0,2] \to I\!R$ be a smooth function such that $\eta = 1$ in $[0,1]$, and $\eta = 0$ in $[\frac{3}{2}, 2]$. We define

$$\tilde{u}_\varepsilon(x) = \eta\left(\frac{d_g(x_\varepsilon, x)}{r_\varepsilon}\right) u_\varepsilon(x) \tag{A7}$$

Given $y \in M$, $\theta \in I\!R$, and $\mu > 0$, we let $V_{(y,\theta,\mu)}$ be the function given by

$$V_{(y,\theta,\mu)}(x) = (1+\theta)\mu^{1-\frac{n}{2}} \eta\left(\frac{d_g(y,x)}{r_\varepsilon}\right) V_0\left(\frac{(1-\varepsilon)^{1/2} d_g(y,x)}{\mu}\right) \tag{A8}$$

where V_0 is as above. For $\varepsilon > 0$ small, we let also Λ_ε be the set of the (y, θ, μ)'s which are such that

$$\frac{d_g(y, x_\varepsilon)}{\mu_\varepsilon} \le 1 \ , \ \frac{1}{2} \le \frac{\mu_\varepsilon}{\mu} \le 2 \ , \ -\frac{1}{2} \le \theta \le \frac{1}{2}$$

We define the functional

$$J_\varepsilon(y,\theta,\mu) = \int_M \left|\nabla(\tilde{u}_\varepsilon - V_{(y,\theta,\mu)})\right|^2 dv_g$$

and let $(y_\varepsilon, \theta_\varepsilon, \overline{\mu}_\varepsilon) \in \Lambda_\varepsilon$ be such that

$$J_\varepsilon(y_\varepsilon, \theta_\varepsilon, \overline{\mu}_\varepsilon) = \min_{(y,\theta,\mu)\in\Lambda_\varepsilon} J_\varepsilon(y,\theta,\mu) \tag{A9}$$

We claim that

$$\frac{d_g(y_\varepsilon, x_\varepsilon)}{\mu_\varepsilon} \to 0 \ , \ \frac{\mu_\varepsilon}{\overline{\mu}_\varepsilon} \to 1 \ , \ \theta_\varepsilon \to 0 \tag{A10}$$

as $\varepsilon \to 0$. In order to prove this claim, we proceed as follows. We know, thanks to $(A4)$, that $J_\varepsilon(x_\varepsilon, 0, \mu_\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, $J_\varepsilon(y_\varepsilon, \theta_\varepsilon, \overline{\mu}_\varepsilon) \to 0$ as $\varepsilon \to 0$. Up to a subsequence, we may assume that

$$\frac{d_g(y_\varepsilon, x_\varepsilon)}{\mu_\varepsilon} \to C_0 \ , \ \frac{\mu_\varepsilon}{\overline{\mu}_\varepsilon} \to C_1 \ , \ \theta_\varepsilon \to \theta_0$$

as $\varepsilon \to 0$. We write that

$$J_\varepsilon(y_\varepsilon,\theta_\varepsilon,\overline{\mu}_\varepsilon) = \int_M |\nabla\tilde{u}_\varepsilon|^2 dv_g + \int_M |\nabla V_{(y_\varepsilon,\theta_\varepsilon,\overline{\mu}_\varepsilon)}|^2 dv_g - 2\int_M \left(\nabla\tilde{u}_\varepsilon, \nabla V_{(y_\varepsilon,\theta_\varepsilon,\overline{\mu}_\varepsilon)}\right) dv_g$$

where $(.,.)$ is the pointwise scalar product with respect to g. In particular,

$$J_\varepsilon(y_\varepsilon,\theta_\varepsilon,\overline{\mu}_\varepsilon) \geq \left(\|\nabla\tilde{u}_\varepsilon\|_2 - \|\nabla V_{(y_\varepsilon,\theta_\varepsilon,\overline{\mu}_\varepsilon)}\|_2\right)^2$$

Noting that $\overline{\mu}_\varepsilon \to 0$ as $\varepsilon \to 0$, it is easy to check that

$$\|\nabla V_{(y_\varepsilon,\theta_\varepsilon,\overline{\mu}_\varepsilon)}\|_2^2 \to (1+\theta_0)^2 K_n^{-1}$$

as $\varepsilon \to 0$. We also have that $\|\nabla\tilde{u}_\varepsilon\|_2^2 \to K_n^{-1}$ as $\varepsilon \to 0$. Hence, $\theta_0 = 0$, and

$$\int_M \left(\nabla\tilde{u}_\varepsilon, \nabla V_{(y_\varepsilon,\theta_\varepsilon,\overline{\mu}_\varepsilon)}\right) dv_g \to \frac{1}{K_n}$$

as $\varepsilon \to 0$. It is easily checked that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_M \left(\nabla\tilde{u}_\varepsilon, \nabla V_{(y_\varepsilon,\theta_\varepsilon,\overline{\mu}_\varepsilon)}\right) dv_g \\
&\quad = \lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{B_{x_\varepsilon}(R\mu_\varepsilon)} \left(\nabla\tilde{u}_\varepsilon, \nabla V_{(y_\varepsilon,\theta_\varepsilon,\overline{\mu}_\varepsilon)}\right) dv_g
\end{aligned}$$

and that

$$\begin{aligned}
&\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{B_{x_\varepsilon}(R\mu_\varepsilon)} \left(\nabla\tilde{u}_\varepsilon, \nabla V_{(y_\varepsilon,\theta_\varepsilon,\overline{\mu}_\varepsilon)}\right) dv_g \\
&\quad = C_1^{\frac{n}{2}-1}\int_{\mathbb{R}^n} \left(\nabla V_0(C_1|x-y_0|), \nabla V_0(|x|)\right) dx
\end{aligned}$$

where $y_0 = \mu_\varepsilon^{-1}\exp_{x_\varepsilon}^{-1}(y_\varepsilon)$. Hence,

$$C_1^{\frac{n}{2}-1}\int_{\mathbb{R}^n} \left(\nabla V_0(C_1|x-y_0|), \nabla V_0(|x|)\right) dx = \frac{1}{K_n}$$

This implies in turn that $C_1 = 1$ and that $y_0 = 0$, so that $(A10)$ is proved. From now on, we let

$$\begin{aligned}
g_\varepsilon(x) &= \exp_{y_\varepsilon}^\star g(r_\varepsilon x) \\
\tilde{v}_\varepsilon(x) &= r_\varepsilon^{\frac{n}{2}-1}\tilde{u}_\varepsilon\left(\exp_{y_\varepsilon}(r_\varepsilon x)\right) \\
v_\varepsilon(x) &= r_\varepsilon^{\frac{n}{2}-1}u_\varepsilon\left(\exp_{y_\varepsilon}(r_\varepsilon x)\right)
\end{aligned}$$

Thanks to $(A10)$, the analysis of subsection 12.3 can be applied to v_ε. In particular, if we let

$$\hat{\mu}_\varepsilon = \frac{\overline{\mu}_\varepsilon}{r_\varepsilon}$$

then $\hat{\mu}_\varepsilon \to 0$ as $\varepsilon \to 0$, and there exists $C > 0$ such that for any x,

$$|x|^{n-2}\hat{\mu}_\varepsilon^{1-\frac{n}{2}} v_\varepsilon(x) \leq C \qquad (A11)$$

Moreover,

$$\lim_{\varepsilon\to 0}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}v_\varepsilon = H \text{ in } C^1_{loc}(\mathbb{R}^n\backslash\{0\}) \tag{A12}$$

where, if $\mathcal{B}=B_0(1)$ is the unit ball in $\mathbb{R}^n$,

$$\begin{aligned} &H(x)=\frac{A_n}{n(n-2)}\left(|x|^{2-n}-1\right)+\frac{A_n}{2n}\left(|x|^2-1\right) \text{ in } \mathcal{B} \\ &H(x)=0 \text{ in } \mathbb{R}^n\backslash\mathcal{B} \end{aligned} \tag{A13}$$

and

$$A_n = n(n-2)2^{n-2}\omega_n^{\frac{2}{n}-1} \tag{A14}$$

In addition,

$$\Delta_{g_\varepsilon}v_\varepsilon + C_\varepsilon\hat{\Sigma}_\varepsilon = \frac{1-\varepsilon}{K_n}v_\varepsilon^{2^\star-1} \tag{A15}$$

where $\hat{\Sigma}_\varepsilon(x)=\Sigma_\varepsilon\left(\exp_{y_\varepsilon}(r_\varepsilon x)\right)$ is such that

$$\lim_{\varepsilon\to 0}\hat{\Sigma}_\varepsilon = 1\!\!I_{\mathcal{B}} \tag{A16}$$

in $L^p_{loc}(\mathbb{R}^n)$ for all $p\geq 1$, $1\!\!I_{\mathcal{B}}$ being the characteristic function of $\mathcal{B}$, and $C_\varepsilon>0$ is such that

$$\lim_{\varepsilon\to 0}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}C_\varepsilon = A_n \tag{A17}$$

Thanks to $(A11)$-$(A13)$ and $(A17)$, for any $R\geq 1$,

$$\lim_{\varepsilon\to 0}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}C_\varepsilon\int_{B_0(R)}v_\varepsilon dv_{g_\varepsilon} = A_n\int_{\mathcal{B}}H dx = \frac{A_n^2\omega_{n-1}}{2n(n+2)} \tag{A18}$$

Now we write that

$$\tilde{v}_\varepsilon(x) = (1+\theta_\varepsilon)\eta(|x|)\hat{\mu}_\varepsilon^{1-\frac{n}{2}}V_0\left(\frac{(1-\varepsilon)^{1/2}|x|}{\hat{\mu}_\varepsilon}\right)+\hat{\mu}_\varepsilon^{\frac{n}{2}-1}w_\varepsilon \tag{A19}$$

where $w_\varepsilon\in C^1_0\left(B_0(2)\right)$, the space of C^1-functions with compact support in $B_0(2)$. Thanks to $(A11)$, $(A12)$ and $(A13)$,

$$\lim_{\varepsilon\to 0}w_\varepsilon = \frac{A_n}{2n}|x|^2-\frac{A_n}{2(n-2)} \tag{A20}$$

in $C^1_{loc}\left(\overline{\mathcal{B}}\backslash\{0\}\right)\cap L^p\left(\mathcal{B}\right)$ for all $p<n/(n-2)$. Independently, the fact that $(y_\varepsilon,\theta_\varepsilon,\overline{\mu}_\varepsilon)$ realizes the infimum of J_ε gives that

$$\begin{aligned} &\int_{B_0(2)}\left(\nabla U_\varepsilon,\nabla w_\varepsilon\right)dv_{g_\varepsilon}=0 \\ &\int_{B_0(2)}\left(\nabla U_{i,\varepsilon},\nabla w_\varepsilon\right)dv_{g_\varepsilon}=0 \\ &\int_{B_0(2)}\left(\nabla \Phi_\varepsilon,\nabla w_\varepsilon\right)dv_{g_\varepsilon}=0 \end{aligned} \tag{A21}$$

where $(.,.)$ is the scalar product with respect to g_ε, and where

$$U_\varepsilon = \eta(|x|)\hat{\mu}_\varepsilon^{1-\frac{n}{2}} V_0\left(\frac{(1-\varepsilon)^{1/2}|x|}{\hat{\mu}_\varepsilon}\right)$$

$$\Phi_\varepsilon = \eta(|x|)\left(-\frac{n-2}{2}V_0\left(\frac{(1-\varepsilon)^{1/2}|x|}{\hat{\mu}_\varepsilon}\right) + V_0'\left(\frac{(1-\varepsilon)^{1/2}|x|}{\hat{\mu}_\varepsilon}\right)\right)$$

$$U_{i,\varepsilon} = \hat{\mu}_\varepsilon^{1-\frac{n}{2}}\frac{\partial}{\partial x_i}\left(\eta(|x|)V_0\left(\frac{(1-\varepsilon)^{1/2}|x|}{\hat{\mu}_\varepsilon}\right)\right)$$

for $i = 1, \dots, n$. A first objective is to compute the L^2-norm of the gradient of w_ε. We start writing that

$$\begin{aligned}&\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{B_0(2)}(\Delta_{g_\varepsilon}\tilde{v}_\varepsilon)w_\varepsilon dv_{g_\varepsilon}\\ &\quad = (1+\theta_\varepsilon)\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{B_0(2)}(\Delta_{g_\varepsilon}U_\varepsilon)w_\varepsilon dv_{g_\varepsilon} + \int_{B_0(2)}|\nabla w_\varepsilon|^2 dv_{g_\varepsilon}\end{aligned}$$

so that, thanks to $(A21)$,

$$\int_{B_0(2)}|\nabla w_\varepsilon|^2 dv_{g_\varepsilon} = \hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{B_0(2)}(\Delta_{g_\varepsilon}\tilde{v}_\varepsilon)w_\varepsilon dv_{g_\varepsilon}$$

Using $(A12)$, $(A13)$, and $(A15)$, it follows that

$$\begin{aligned}&\int_{B_0(2)}|\nabla w_\varepsilon|^2 dv_{g_\varepsilon}\\ &\quad = \frac{1-\varepsilon}{K_n}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}} v_\varepsilon^{2^\star-1}w_\varepsilon dv_{g_\varepsilon} - C_\varepsilon\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{\Sigma}_\varepsilon w_\varepsilon dv_{g_\varepsilon} + o(1)\end{aligned}$$

Thanks to $(A16)$, $(A17)$ and $(A20)$ we then get that

$$\int_{B_0(2)}|\nabla w_\varepsilon|^2 dv_{g_\varepsilon} = \frac{1-\varepsilon}{K_n}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}} v_\varepsilon^{2^\star-1}w_\varepsilon dv_{g_\varepsilon} + \frac{2A_n^2\omega_{n-1}}{n(n-2)(n+2)} + o(1) \tag{A22}$$

Since $n = 4, 5$, we can write that

$$\begin{aligned}&\int_{\mathcal{B}} v_\varepsilon^{2^\star-1}w_\varepsilon dv_{g_\varepsilon} = (1+\theta_\varepsilon)^{2^\star-1}\int_{\mathcal{B}} U_\varepsilon^{2^\star-1}w_\varepsilon dv_{g_\varepsilon}\\ &+(2^\star-1)(1+\theta_\varepsilon)^{2^\star-2}\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}w_\varepsilon^2 dv_{g_\varepsilon}\\ &+O\left(\hat{\mu}_\varepsilon^{n-2}\int_{\mathcal{B}} U_\varepsilon^{2^\star-3}|w_\varepsilon|^3 dv_{g_\varepsilon}\right) + O\left(\hat{\mu}_\varepsilon^{\frac{n}{2}+1}\int_{\mathcal{B}}|w_\varepsilon|^{2^\star}dv_{g_\varepsilon}\right)\end{aligned}$$

Since U_ε is radially symmetrical,

$$\Delta_{g_\varepsilon}U_\varepsilon = \frac{1-\varepsilon}{K_n}U_\varepsilon^{2^\star-1} + O\left(r_\varepsilon|x|U_\varepsilon'(|x|)\right)$$

in $\mathcal{B}$. Therefore, using $(A21)$,

$$\begin{aligned}
&\int_{\mathcal{B}} U_\varepsilon^{2^\star-1} w_\varepsilon dv_{g_\varepsilon}\\
&= \frac{K_n}{1-\varepsilon}\int_{\mathcal{B}} (\Delta_{g_\varepsilon} U_\varepsilon) w_\varepsilon dv_{g_\varepsilon} + O\left(r_\varepsilon \int_{\mathcal{B}} |x||\nabla U_\varepsilon||w_\varepsilon| dv_{g_\varepsilon}\right)\\
&= -\frac{K_n}{1-\varepsilon}\int_{B_0(2)\setminus\mathcal{B}} (\Delta_{g_\varepsilon} U_\varepsilon) w_\varepsilon dv_{g_\varepsilon} + O\left(r_\varepsilon \int_{\mathcal{B}} |x||\nabla U_\varepsilon||w_\varepsilon| dv_{g_\varepsilon}\right)
\end{aligned}$$

Independently, thanks to Hölder's inequality, and to the Euclidean Sobolev inequality, we can write that

$$\int_{\mathcal{B}} |x||\nabla U_\varepsilon||w_\varepsilon| dv_{g_\varepsilon} \le C\|\nabla w_\varepsilon\|_2 \left(\int_{\mathcal{B}} \left(|x||\nabla U_\varepsilon|\right)^{\frac{2^\star}{2^\star-1}} dv_{g_\varepsilon}\right)^{\frac{2^\star-1}{2^\star}}$$

where $C>0$ is independent of ε. Since $n=4,5$,

$$\int_{\mathcal{B}} \left(|x||\nabla U_\varepsilon|\right)^{\frac{2^\star}{2^\star-1}} dv_{g_\varepsilon} = O\left(\hat{\mu}_\varepsilon^{\frac{n(n-2)}{n+2}}\right)$$

Therefore,

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-1} w_\varepsilon dv_{g_\varepsilon} = O\left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\right) + o\left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\|\nabla w_\varepsilon\|_2\right)$$

Coming back to $(A22)$ we then get that

$$\begin{aligned}
&(1+o(1))\int_{B_0(2)} |\nabla w_\varepsilon|^2 dv_{g_\varepsilon}\\
&\quad = O(1) + (2^\star-1)(1+\theta_\varepsilon)^{2^\star-2}(1-\varepsilon)\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dv_{g_\varepsilon}
\end{aligned} \tag{A23}$$

We claim now that

$$\int_{B_0(2)} |\nabla w_\varepsilon|^2 dv_{g_\varepsilon} = O(1) \tag{A24}$$

In order to prove $(A24)$, we proceed by contradiction, assuming that

$$\int_{B_0(2)} |\nabla w_\varepsilon|^2 dv_{g_\varepsilon} \to +\infty \quad\text{and}\quad \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dv_{g_\varepsilon} \to +\infty$$

as $\varepsilon \to 0$, and we consider the following eigenvalue problem:

$$\begin{aligned}
&\Delta_{g_\varepsilon}\varphi_{i,\varepsilon} = \mu_{i,\varepsilon} U_\varepsilon^{2^\star-2}\varphi_{i,\varepsilon} \text{ in } B_0(2)\\
&\varphi_{i,\varepsilon} = 0 \text{ on } \partial B_0(2)
\end{aligned}$$

where

$$\int_{B_0(2)} U_\varepsilon^{2^\star-2}\varphi_{i,\varepsilon}\varphi_{j,\varepsilon} dv_{g_\varepsilon} = \delta_{ij}$$

and $\mu_{1,\varepsilon} \le \ldots \le \mu_{i,\varepsilon} \le \ldots$. Since $g_\varepsilon \to \xi$ in $C^0_{loc}(I\!R^n)$ as $\varepsilon \to 0$, ξ the Euclidean metric, the analysis of subsection 12.1.6 can be applied to the present situation. We then get that for any $i \ge 1$,

$$\begin{aligned}
&\mu_{i,\varepsilon} \to \mu_i \text{ , and}\\
&\int_{B_0(2)} U_\varepsilon^{2^\star-2}\left(\varphi_{i,\varepsilon} - \eta\psi_{i,\varepsilon}\right) dv_{g_\varepsilon} \to 0
\end{aligned}$$

when $\varepsilon \to 0$, where

$$\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\psi_{i,\varepsilon}\left(\frac{\hat{\mu}_\varepsilon|x|}{\sqrt{1-\varepsilon}}\right) = \psi_i(x)$$

and

$$\Delta\psi_i = \mu_i V_0(|x|)^{2^\star-2}\psi_i \text{ in } I\!R^n ,$$
$$\int_{I\!R^n} V_0(|x|)^{2^\star-2}\psi_i^2 dx < +\infty$$

Thanks to Bianchi-Egnell [4] and Rey [34],

$$\mu_1 = \frac{1}{K_n} \ , \ \mu_2 = \ldots = \mu_{n+2} = \frac{2^\star-1}{K_n} \ , \ \mu_{n+3} > \frac{2^\star-1}{K_n}$$

and $\psi_1(x) = V_0(|x|)$ while

$$\psi_i(x) = \left(1 + \frac{\omega_n^{2/n}}{4}|x|^2\right)^{-\frac{n}{2}} x_{i-1} \text{ for } i = 2,\ldots,n+1 ,$$
$$\psi_{n+2}(x) = \left(1 + \frac{\omega_n^{2/n}}{4}|x|^2\right)^{-\frac{n}{2}}\left(1 - \frac{\omega_n^{2/n}}{4}|x|^2\right)$$

We let

$$w_\varepsilon = \sum_{i=1}^{n+2}\alpha_{i,\varepsilon}\varphi_{i,\varepsilon} + R_\varepsilon$$

where

$$\alpha_{i,\varepsilon} = \frac{\int_{B_0(2)}(\nabla w_\varepsilon, \nabla\varphi_{i,\varepsilon})\, dv_{g_\varepsilon}}{\int_{B_0(2)}|\nabla\varphi_{i,\varepsilon}|^2 dv_{g_\varepsilon}}$$

Hence,

$$\alpha_{i,\varepsilon} = \frac{1}{\mu_{i,\varepsilon}}\int_{B_0(2)}(\nabla w_\varepsilon, \nabla(\varphi_{i,\varepsilon} - \eta\psi_{i,\varepsilon}))\, dv_{g_\varepsilon} + \frac{1}{\mu_{i,\varepsilon}}\int_{B_0(2)}(\nabla w_\varepsilon, \nabla(\eta\psi_{i,\varepsilon}))\, dv_{g_\varepsilon}$$

and it is easily checked that this implies that

$$\alpha_{i,\varepsilon}^2 = o\left(\|\nabla w_\varepsilon\|_2^2\right) + O\left(X_\varepsilon^2\right)$$

where

$$X_\varepsilon = \int_{B_0(2)}(\nabla w_\varepsilon, \nabla(\eta\psi_{i,\varepsilon}))\, dv_{g_\varepsilon}$$

Thanks to $(A21)$ we then get that for any $i = 1,\ldots,n+2$,

$$\alpha_{i,\varepsilon}^2 = o\left(\|\nabla w_\varepsilon\|_2^2\right) + o(1)$$

Independently,

$$\int_{B_0(2)}|\nabla w_\varepsilon|^2 dv_{g_\varepsilon} \geq \sum_{i=1}^{n+2}\mu_{i,\varepsilon}\alpha_{i,\varepsilon}^2 + \mu_{n+3,\varepsilon}\int_{B_0(2)} U_\varepsilon^{2^\star-2}R_\varepsilon^2 dv_{g_\varepsilon}$$

and

$$\int_{B_0(2)} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dv_{g_\varepsilon} = \sum_{i=1}^{n+2} \alpha_{i,\varepsilon}^2 + \int_{B_0(2)} U_\varepsilon^{2^\star-2} R_\varepsilon^2 dv_{g_\varepsilon}$$

Therefore, since $\mu_{n+3,\varepsilon} \to \mu_{n+3}$ as $\varepsilon \to 0$, and $\mu_{n+3} > \frac{2^\star-1}{K_n}$,

$$\liminf_{\varepsilon\to 0} \frac{\int_{B_0(2)} |\nabla w_\varepsilon|^2 dv_{g_\varepsilon}}{\int_{B_0(2)} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dv_{g_\varepsilon}} > \frac{2^\star - 1}{K_n}$$

Noting that

$$\lim_{\varepsilon\to 0} \frac{\int_{B_0(2)} |\nabla w_\varepsilon|^2 dv_{g_\varepsilon}}{\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dv_{g_\varepsilon}} = 1$$

it follows that

$$\liminf_{\varepsilon\to 0} \frac{\int_{B_0(2)} |\nabla w_\varepsilon|^2 dv_{g_\varepsilon}}{\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dv_{g_\varepsilon}} > \frac{2^\star - 1}{K_n}$$

and we get a contradiction by coming back to $(A23)$. This proves $(A24)$. Now we compute

$$A_\varepsilon = \int_{B_0(2)} |\nabla \tilde{v}_\varepsilon|^2 dx$$

We let $\tilde{\eta}_\varepsilon$ be the function given by $\tilde{\eta}_\varepsilon(x) = \eta\left(r_\varepsilon^{-1} d_g(x_\varepsilon, \exp_{y_\varepsilon}(r_\varepsilon x)\right)$. Then, on the one hand,

$$\begin{aligned}\int_{B_0(2)} |\nabla \tilde{v}_\varepsilon|^2 dv_{g_\varepsilon} &= \int_{B_0(2)} \tilde{\eta}_\varepsilon^2 v_\varepsilon \Delta_{g_\varepsilon} v_\varepsilon dv_{g_\varepsilon} + \int_{B_0(2)} |\nabla \tilde{\eta}_\varepsilon|^2 v_\varepsilon^2 dv_{g_\varepsilon} \\ &= \frac{1-\varepsilon}{K_n} \int_{B_0(2)} \tilde{\eta}_\varepsilon^2 v_\varepsilon^{2^\star} dv_{g_\varepsilon} - C_\varepsilon \int_{B_0(2)} \tilde{\eta}_\varepsilon^2 v_\varepsilon dv_{g_\varepsilon} + \int_{B_0(2)} |\nabla \tilde{\eta}_\varepsilon|^2 v_\varepsilon^2 dv_{g_\varepsilon}\end{aligned}$$

Thanks to $(A5)$, but also $(A12)$, $(A13)$, and $(A18)$, it follows that

$$\int_{B_0(2)} |\nabla \tilde{v}_\varepsilon|^2 dv_{g_\varepsilon} = \frac{1-\varepsilon}{K_n} - \frac{A_n^2 \omega_{n-1}}{2n(n+2)} \hat{\mu}_\varepsilon^{n-2} + o\left(\hat{\mu}_\varepsilon^{n-2}\right)$$

On the other hand, thanks to $(A21)$,

$$\begin{aligned}\int_{B_0(2)} |\nabla \tilde{v}_\varepsilon|^2 dv_{g_\varepsilon} &= (1+\theta_\varepsilon)^2 \int_{B_0(2)} |\nabla U_\varepsilon|^2 dv_{g_\varepsilon} + \hat{\mu}_\varepsilon^{n-2} \int_{B_0(2)} |\nabla w_\varepsilon|^2 dv_{g_\varepsilon} \\ &\quad +2(1+\theta_\varepsilon)\hat{\mu}_\varepsilon^{\frac{n}{2}-1} \int_{B_0(2)} (\nabla U_\varepsilon, \nabla w_\varepsilon)\, dv_{g_\varepsilon} \\ &= (1+\theta_\varepsilon)^2 \int_{B_0(2)} |\nabla U_\varepsilon|^2 dv_{g_\varepsilon} + \hat{\mu}_\varepsilon^{n-2} \int_{B_0(2)} |\nabla w_\varepsilon|^2 dv_{g_\varepsilon}\end{aligned}$$

Noting that U_ε is radially symmetrical, thanks to the Cartan expansion of a metric in geodesic normal coordinates, it is easily checked that

$$\int_{B_0(2)} |\nabla U_\varepsilon|^2 dv_{g_\varepsilon} = \int_{B_0(2)} |\nabla U_\varepsilon|^2 dx - \frac{(n-2)(n+2)}{6(n-4)(1-\varepsilon)^{n/2}} S_g(y_\varepsilon) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o\left(\hat{\mu}_\varepsilon^{n-2}\right)$$

when $n = 5$, and that

$$\int_{B_0(2)} |\nabla U_\varepsilon|^2 dv_{g_\varepsilon} = \int_{B_0(2)} |\nabla U_\varepsilon|^2 dx - \frac{8\omega_3}{3\omega_4(1-\varepsilon)^2} S_g(y_\varepsilon) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o\left(\hat{\mu}_\varepsilon^2\right)$$

when $n = 4$. Combining the above quations, it follows that

$$\begin{aligned}
&(1+\theta_\varepsilon)^2 \int_{B_0(2)} |\nabla U_\varepsilon|^2 dx + \hat{\mu}_\varepsilon^{n-2} \int_{B_0(2)} |\nabla w_\varepsilon|^2 dv_{g_\varepsilon} \\
&= \frac{1-\varepsilon}{K_n} - \frac{A_n^2 \omega_{n-1}}{2n(n+2)} \hat{\mu}_\varepsilon^{n-2} + \frac{(n-2)(n+2)(1+\theta_\varepsilon)^2}{6(n-4)(1-\varepsilon)^{n/2}} S_g(y_\varepsilon) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o\left(\hat{\mu}_\varepsilon^{n-2}\right)
\end{aligned}$$

when $n = 5$, and that

$$\begin{aligned}
&(1+\theta_\varepsilon)^2 \int_{B_0(2)} |\nabla U_\varepsilon|^2 dx + \hat{\mu}_\varepsilon^{n-2} \int_{B_0(2)} |\nabla w_\varepsilon|^2 dv_{g_\varepsilon} \\
&= \frac{1-\varepsilon}{K_4} - \frac{A_4^2 \omega_3}{48} \hat{\mu}_\varepsilon^2 + \frac{8\omega_3 (1+\theta_\varepsilon)^2}{3\omega_4(1-\varepsilon)^2} S_g(y_\varepsilon) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o\left(\hat{\mu}_\varepsilon^2\right)
\end{aligned}$$

when $n = 4$. Coming back to the computation of A_ε, we can write that

$$\begin{aligned}
&\int_{B_0(2)} |\nabla \tilde{v}_\varepsilon|^2 dx = (1+\theta_\varepsilon)^2 \int_{B_0(2)} |\nabla U_\varepsilon|^2 dx \\
&+2(1+\theta_\varepsilon) \hat{\mu}_\varepsilon^{\frac{n}{2}-1} \int_{B_0(2)} (\nabla U_\varepsilon, \nabla w_\varepsilon)\, dx + \hat{\mu}_\varepsilon^{n-2} (1+o(1)) \int_{B_0(2)} |\nabla w_\varepsilon|^2 dv_{g_\varepsilon}
\end{aligned}$$

Thanks to $(A21)$, and to the Cartan expansion of a metric in geodesic normal coordinates,

$$\begin{aligned}
\int_{B_0(2)} (\nabla U_\varepsilon, \nabla w_\varepsilon)\, dx &= O\left(r_\varepsilon^2 \int_{B_0(2)} |x|^2 |\nabla U_\varepsilon| |\nabla w_\varepsilon| dx\right) \\
&= O\left(r_\varepsilon^2 \hat{\mu}_\varepsilon^{\frac{n}{2}-1} \|\nabla w_\varepsilon\|_2\right) \\
&= o\left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\right) + o\left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1} \|\nabla w_\varepsilon\|_2^2\right)
\end{aligned}$$

Therefore, thanks to $(A24)$, and the above equations,

$$A_\varepsilon = \frac{1-\varepsilon}{K_n} - \frac{A_n^2 \omega_{n-1}}{2n(n+2)} \hat{\mu}_\varepsilon^{n-2} + \frac{(n-2)(n+2)(1+\theta_\varepsilon)^2}{6(n-4)(1-\varepsilon)^{n/2}} S_g(y_\varepsilon) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o\left(\hat{\mu}_\varepsilon^{n-2}\right) \qquad (A25)$$

when $n = 5$, and

$$A_\varepsilon = \frac{1-\varepsilon}{K_4} - \frac{A_4^2 \omega_3}{48} \hat{\mu}_\varepsilon^2 + \frac{8\omega_3 (1+\theta_\varepsilon)^2}{3\omega_4(1-\varepsilon)^2} S_g(y_\varepsilon) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o\left(\hat{\mu}_\varepsilon^2\right) \qquad (A26)$$

when $n = 4$. Now we compute

$$B_\varepsilon = \left(\int_{B_0(2)} \tilde{v}_\varepsilon^{2^\star} dx\right)^{2/2^\star}$$

Thanks to $(A24)$, we can write that

$$\begin{aligned}\int_{B_0(2)} \tilde{v}_\varepsilon^{2^\star} dx &= (1+\theta_\varepsilon)^{2^\star} \int_{B_0(2)} U_\varepsilon^{2^\star} dx + 2^\star (1+\theta_\varepsilon)^{2^\star-1} \hat{\mu}_\varepsilon^{\frac{n}{2}-1} \int_{B_0(2)} U_\varepsilon^{2^\star-1} w_\varepsilon dx \\ &\quad + \frac{2^\star(2^\star-1)}{2} (1+\theta_\varepsilon)^{2^\star-2} \hat{\mu}_\varepsilon^{n-2} \int_{B_0(2)} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dx + o\left(\hat{\mu}_\varepsilon^{n-2}\right)\end{aligned}$$

Thanks to $(A23)$-$(A24)$ we can also write that

$$\begin{aligned}&\int_{B_0(2)} \tilde{v}_\varepsilon^{2^\star} dv_{g_\varepsilon} = 1 + o\left(\hat{\mu}_\varepsilon^{n-2}\right) \\ &= (1+\theta_\varepsilon)^{2^\star} \int_{B_0(2)} U_\varepsilon^{2^\star} dv_{g_\varepsilon} + 2^\star (1+\theta_\varepsilon)^{2^\star-1} \hat{\mu}_\varepsilon^{\frac{n}{2}-1} \int_{B_0(2)} U_\varepsilon^{2^\star-1} w_\varepsilon dv_{g_\varepsilon} \\ &+ \frac{2^\star(2^\star-1)}{2} (1+\theta_\varepsilon)^{2^\star-2} \hat{\mu}_\varepsilon^{n-2} \int_{B_0(2)} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dx + o\left(\hat{\mu}_\varepsilon^{n-2}\right)\end{aligned}$$

The Cartan expansion of a metric in geodesic normal coordinates gives that

$$\int_{B_0(2)} U_\varepsilon^{2^\star} dv_{g_\varepsilon} = \int_{B_0(2)} U_\varepsilon^{2^\star} dx - \frac{nK_n}{6} S_g(y_\varepsilon)(1-\varepsilon)^{-1-\frac{n}{2}} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o\left(\hat{\mu}_\varepsilon^{n-2}\right)$$

On the other hand, it is easily checked that

$$\int_{B_0(2)} U_\varepsilon^{2^\star-1} w_\varepsilon dv_{g_\varepsilon} = \int_{B_0(2)} U_\varepsilon^{2^\star-1} w_\varepsilon dx + o\left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\right)$$

Combining the above equations, we get that

$$\int_{B_0(2)} \tilde{v}_\varepsilon^{2^\star} dx = 1 + (1+\theta_\varepsilon)^{2^\star} \frac{nK_n}{6} S_g(y_\varepsilon)(1-\varepsilon)^{-1-\frac{n}{2}} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o\left(\hat{\mu}_\varepsilon^{n-2}\right)$$

and it follows that

$$B_\varepsilon = 1 + \frac{(n-2)K_n}{6} S_g(y_\varepsilon)(1-\varepsilon)^{-1-\frac{n}{2}} (1+\theta_\varepsilon)^{2^\star} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o\left(\hat{\mu}_\varepsilon^{n-2}\right) \tag{A27}$$

The sharp Euclidean Sobolev inequality applied to the $\tilde{v}_\varepsilon$'s reads as

$$B_\varepsilon \leq K_n A_\varepsilon \tag{A28}$$

Combining $(A25)$-$(A28)$, we get that

$$\varepsilon + \frac{A_4^2 K_4 \omega_3}{48} \hat{\mu}_\varepsilon^2 \leq \frac{8K_4\omega_3(1+\theta_\varepsilon)^2}{3\omega_4(1-\varepsilon)^2} S_g(y_\varepsilon) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o\left(\hat{\mu}_\varepsilon^2\right) \tag{A29}$$

when $n = 4$, and that

$$\begin{aligned}&\varepsilon + \frac{(n-2)K_n(1+\theta_\varepsilon)^2}{6(1-\varepsilon)^{\frac{n}{2}}} \left(\frac{(1+\theta_\varepsilon)^{2^\star-2}}{1-\varepsilon} - \frac{n+2}{n-4}\right) S_g(y_\varepsilon) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 \\ &\leq -\frac{A_n^2 K_n \omega_{n-1}}{2n(n+2)} \hat{\mu}_\varepsilon^{n-2} + o\left(\hat{\mu}_\varepsilon^{n-2}\right)\end{aligned} \tag{A30}$$

when $n = 5$. Since $S_g(y_\varepsilon) \leq 0$, $(A29)$ and $(A30)$ are impossible. This is the contradiction we were looking for. Theorem 4.3 is proved.

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