Manifolds which are close to space forms and the Bemelmans, Min-Oo and Ruh smoothing effect of the Ricci flow

by

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A compact Riemannian manifold (M, g) is said to be ε-flat if its sectional curvature is bounded in terms of its diameter in the following way:

$$|K_g| \leq \varepsilon d_g(M)^{-2}$$

where K_g is the sectional curvature of g, and $d_g(M)$ is the diameter of (M, g). In [8], Gromov stated the following beautiful result: for

$$\varepsilon \leq \varepsilon_n = \exp\Big(-\exp\Big(\exp(n^2)\Big)\Big)$$

any ε-flat n-manifold is covered by a nilmanifold (i.e. a compact quotient of a nilpotent Lie group). In [3], Buser and Karcher gave a detailed proof of this result of Gromov. They obtained as a by-product of their proof that for $\varepsilon \leq \varepsilon_n$, any ε-flat n-manifold (M, g) which satisfies that

$$i_g(M) > 2^{-n^3}\sqrt{\frac{\varepsilon}{\varepsilon_n}}d_g(M)$$

is covered by a torus (corollary 1.5.1 of [3]), where $i_g(M)$ is the injectivity radius of (M, g). Even if we forget about effectiveness, the ε-dependency in the estimate relating the injectivity radius and the diameter is difficult to obtain. On the other hand, as shown below, if we replace this estimate by an estimate like $i_g(M) \geq \alpha d_g(M)$, $\alpha > 0$ independent of ε, then the argument (in its noneffective part) simplifies considerably. Moreover, the result easily extends to manifolds which are close to space forms (in a suitable sense). We show in this short and informal note how one can get such extensions with very simple arguments.

Let Rm_g be the Riemann curvature of g. By definition, the concircular curvature Z_g is the $(4,0)$-tensor field given by the relation

$$Z_g = Rm_g - \frac{S_g}{2n(n-1)} g \odot g$$

where n is the dimension of M, S_g is the scalar curvature of g, and $\odot$ is the Kulkarni-Nomizu product. It is easily seen that a Riemannian manifold (M,g) is a space form (i.e. has constant sectional curvature) if and only if $Z_g \equiv 0$. There are several possible extensions of the notion of an ε-flat manifold. We propose here the following extension:

Definition 0.1 *Let $\varepsilon > 0$ and K be given. A compact Riemannian manifold (M,g) is said to be ε-close to a K-space form if*

$$|Z_g| + \left| S_g - K d_g(M)^{-2} \right| \leq \varepsilon d_g(M)^{-2}$$

where Z_g is the concircular curvature of g, S_g is its scalar curvature, and $d_g(M)$ is the diameter of M with respect to g.

It is easily checked that there exists $\varepsilon' = \varepsilon(n,\varepsilon)$, satisfying that $\varepsilon' \to 0$ as $\varepsilon \to 0$, such that (M,g) is ε-flat if and only if (M,g) is ε'-close to a 0-space form in the sense of the definition above. We prove the following result in this note:

Theorem 0.1 *Let n, $\alpha > 0$ and K be given. There exists $\varepsilon = \varepsilon(n,\alpha,K)$, $\varepsilon > 0$, such that any compact Riemannian n-manifold (M,g) which is ε-close to a K-space form and which satisfies $i_g(M) \geq \alpha d_g(M)$, is diffeomorphic to a space form of constant curvature K.*

Let $V_g(M)$ be the volume of M with respect to g. We know from Cheeger-Gromov-Taylor [5] that under the bound $|K_g| \leq \Lambda$, $\Lambda > 0$ arbitrary, the existence of $i > 0$ and $V > 0$ such that $i_g(M) \geq i$ and $V_g(M) \leq V$ is equivalent to the existence of $v > 0$ and $d > 0$ such that $V_g(M) \geq v$ and $d_g(M) \leq d$. It easily follows that we can replace the condition $i_g(M) \geq \alpha d_g(M)$ by $V_g(M) \geq \alpha d_g(M)^n$ in Theorem 0.1. Note that by taking $K = 0$ in Theorem 0.1, we recover a kind of noneffective version of the Buser-Karcher result mentioned above. Recall that by the Bieberbach theorem, a manifold which possesses a flat metric is covered by a torus.

We prove Theorem 0.1 in what follows. The theorem can be thought as a consequence of the work of Nikolaev [12]. We give here an alternative easy proof of the theorem which, as a by-product, turns out to illustrate the very nice smoothing effect of the Ricci flow pointed out in Bemelmans, Min-Oo and Ruh [2]. Given n integer, and $i, A > 0$, let $\mathcal{R}(n,i,d)$ be the class of compact Riemannian n-manifolds (M,g) which are such that $|Rm_g| \leq 1$, $i_g(M) \geq i$, and $d_g(M) = d$. We proceed by contradiction. Up to rescaling, we get the existence of a sequence (M_k, g_k) in $\mathcal{R}(n,i,d)$ for some n, i, d as above, such that

$$\lim_{k \to +\infty} |Z_{g_k}| = 0 \tag{1}$$

and

$$\lim_{k \to +\infty} S_{g_k} = \overline{S} \tag{2}$$

where $\overline{S} = cK$ for some positive constant c independent of k. We know from the Anderson-Cheeger-Gromov convergence theorem, we refer to Anderson [1], Cheeger [4] and Gromov [9] for more details, that a subsequence of (M_k, g_k) converges in the $C^{1,\theta}$-topology to a Riemannian manifold (M_0, g_0). If the convergence was C^2, we would immediatly conclude that g_0 has constant curvature K_0, with $n(n-1)K_0 = \overline{S}$, and the theorem would be proved. In order to avoid this (little) difficulty that the convergence is only $C^{1,\theta}$, we use the smoothing operator introduced by Bemelmans, Min-Oo and Ruh [2]. Let $\mathcal{M}$ be the class of compact Riemannian n-manifolds (M, g) which are such that $|Rm_g| \leq 1$. According to Bemelmans, Min-Oo and Ruh [2], for any $\varepsilon > 0$, there exists a smoothing operator $\mathcal{S}_\varepsilon : \mathcal{M} \to \mathcal{M}$, and there exists a positive constant $C_1 = C_1(n, \varepsilon)$, such that for any $(M, g) \in \mathcal{M}$, $\mathcal{S}_\varepsilon\left((M, g)\right) = (M, S_\varepsilon(g))$ and the following holds:

$$\|S_\varepsilon(g) - g\|_{C^0} \leq \varepsilon \tag{3}$$

$$\left\|\nabla^j Rm_{S_\varepsilon(g)}\right\|_{C^0} \leq C_1 \|Rm_g\|_{C^0} \quad \text{for all } j = 0, 1 \tag{4}$$

The smoothing operator is constructed with the help of Hamilton's Ricci flow [10]. Hamilton's Ricci flow reads as

$$\frac{\partial g}{\partial t} = -2Rc_g$$

Up to a constant arbitrarily close to 1 (we refer to [2] for details), $S_\varepsilon(g) = g_T$ for some T close to 0 suitably chosen. It is easily seen that the following evolution inequality holds for the concircular curvature (a possible reference is Huisken [11]):

$$\frac{\partial |Z_g|^2}{\partial t} + \Delta_g |Z_g|^2 \leq 20 |Rm_g| |Z_g|^2$$

where $\Delta_g = -div_g \nabla$. In the same order of ideas, the evolution equation for the scalar curvature is

$$\frac{\partial S_g}{\partial t} = -\Delta_g S_g + 2|Rc_g|^2$$

where Rc_g is the Ricci curvature of g, while $\frac{\partial}{\partial t} dv_g = -S_g dv_g$. In particular,

$$\frac{d}{dt}\int_M S_g dv_g = 2\int_M |Rc_g|^2 dv_g - \int_M S_g^2 dv_g$$

It is then easy to check from the construction in Bemelmans, Min-Oo and Ruh [2] that there exist positive constants $C_2 = C_2(n, \varepsilon)$ and $C_3 = C_3(n, \varepsilon)$ such that

$$\left\|Z_{S_\varepsilon(g)}\right\|_{C^0} \leq C_2 \|Z_g\|_{C^0} \tag{5}$$

and

$$\left|\int_M S_{S_\varepsilon(g)} dv_{S_\varepsilon(g)} - \int_M S_g dv_g\right| \leq C_3 \varepsilon \tag{6}$$

We let $g_{\varepsilon,k} = S_\varepsilon(g_k)$. A simple consequence of the estimates obtained by Anderson [1] for the harmonic radius, and of (3), is that there exists $Q > 1$ such that for ε sufficiently small, and any k,

$$Q^{-1} g_k \leq g_{\varepsilon,k} \leq Q g_k \tag{7}$$

in the sense of bilinear forms. An elementary packing argument as in Croke [6] shows that under the bounds $Rc_g \geq \lambda$ and $i_g(M) \geq i$, a bound like $V_g(M) \leq V$ is equivalent to a bound like $d_g(M) \leq d$. It follows from this result, from the Cheeger-Gromov-Taylor estimate [5], and from (7) that the diameter of $(M_k, g_{\varepsilon,k})$ is uniformly bounded from above, and that the volume of $(M_k, g_{\varepsilon,k})$ is uniformly bounded from below. Summarizing, taking into account (4), (5) and (6), the following holds:

$$V_{g_{\varepsilon,k}}(M_k) \geq v \text{ and } d_{g_{\varepsilon,k}} \leq D; \tag{8}$$

$$\left\|\nabla^j Rm_{g_{\varepsilon,k}}\right\|_{C^0} \leq C_4 \text{ for any } j = 0, 1; \tag{9}$$

$$\lim_{k\to+\infty} \left\|Z_{g_{\varepsilon,k}}\right\|_{C^0} = 0; \tag{10}$$

$$\limsup_{k\to+\infty} \left|\int_{M_k} S_{g_{\varepsilon,k}} dv_{g_{\varepsilon,k}} - \int_{M_k} S_{g_k} dv_{g_k}\right| \leq C_3\varepsilon \tag{11}$$

where v, D, and C_4 are positive constant which are independent of k. By (8) and (9), the Anderson-Cheeger-Gromov convergence theorem yields the existence of a smooth compact Riemannian manifold M_ε, the existence of a $C^{2,\theta}$-Riemannian metric g_ε on M_ε, and the existence of $C^{3,\theta}$-diffeomorphisms $\Phi_{\varepsilon,k} : M_\varepsilon \to M_k$ such that in any chart of the smooth complete atlas of M_ε, the components of $\Phi^\star_{\varepsilon,k} g_{\varepsilon,k}$ converge in $C^{2,\theta}_{loc}$ to the components of g_ε as $k \to +\infty$. By (10), $Z_{g_\varepsilon} \equiv 0$, so that g_ε has constant sectional curvature K_ε. In particular, g_ε is Einstein, and by DeTurck-Kazdan [7], M_ε possesses a smooth atlas, C^3-compatible with the original structure, such that g_ε is smooth in this atlas. Since $g_{\varepsilon,k} \in \mathcal{M}$, K_ε is uniformly bounded with respect to ε. By (8), we also get a uniform lower bound for $V_{g_\varepsilon}(M_\varepsilon)$, and a uniform upper bound for $d_{g_\varepsilon}(M_\varepsilon)$. Applying once more the Anderson-Cheeger-Gromov convergence theorem, a subsequence $(M_\varepsilon, g_\varepsilon)$ converges in the $C^{2,\theta}$-topology to some space form (M_0, g_0) in the following sense: there exists a space form (M_0, g_0), and there exists $C^{3,\theta}$-diffeomorphisms $\Phi_\varepsilon : M_0 \to M_\varepsilon$ such that in any chart of the smooth complete atlas of M_0, the components of $\Phi^\star_\varepsilon g_\varepsilon$ converge in $C^{2,\theta}_{loc}$ to the components of g_0 as $\varepsilon \to 0$. We set $\Psi_{\varepsilon,k} = \Phi_{\varepsilon,k} \circ \Phi_\varepsilon$. Then,

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{k\to+\infty} \int_{M_k} S_{g_{\varepsilon,k}} dv_{g_{\varepsilon,k}} &= \lim_{\varepsilon\to 0}\lim_{k\to+\infty} \int_{M_0} S_{\Psi^\star_{\varepsilon,k} g_{\varepsilon,k}} dv_{\Psi^\star_{\varepsilon,k} g_{\varepsilon,k}} \\ &= \int_{M_0} S_{g_0} dv_{g_0}\end{aligned}$$

while it follows from (3) and the construction of $g_{\varepsilon,k}$ that for any sequence $(f_{\varepsilon,k})$ of continuous functions on M_k,

$$\lim_{\varepsilon\to 0}\lim_{k\to+\infty} \left|\int_{M_k} f_{\varepsilon,k} dv_{g_{\varepsilon,k}} - \int_{M_k} f_{\varepsilon,k} dv_{g_k}\right| = 0 \tag{12}$$

provided that the $f_{\varepsilon,k}$'s are uniformly bounded in ε and k. By (2),

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{k\to+\infty} \int_{M_k} S_{g_k} dv_{g_{\varepsilon,k}} &= \lim_{\varepsilon\to 0}\lim_{k\to+\infty} \int_{M_0} S_{\Psi^\star_{\varepsilon,k} g_k} dv_{\Psi^\star_{\varepsilon,k} g_k} \\ &= \int_{M_0} \overline{S} dv_{g_0}\end{aligned}$$

Coming back to (11), we then get with (12) that

$$\int_{M_0} \left(S_{g_0} - \overline{S}\right) dv_{g_0} = 0$$

and since S_{g_0} is constant, we must have that $S_{g_0} = \overline{S}$. In particular, g_0 has constant sectional curvature K_0 such that $n(n-1)K_0 = \overline{S}$. Noting that two smooth manifolds which are C^1-diffeomorphic are also C^∞-diffeomorphic, this proves Theorem 0.1.

Theorem 0.1 still holds if we replace the condition in Definition 0.1 by the condition

$$|Z_g| + \left|S_g - KV_g(M)^{-2/n}\right| \leq \varepsilon V_g(M)^{-2/n}$$

Almost no changes in the proof are needed. As above, the condition $i_g(M) \geq \alpha d_g(M)$ can also be replaced by the condition $V_g(M) \geq \alpha d_g(M)^n$.

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