Stationary Kirchhoff equations with powers

by

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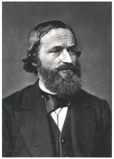
1. The Kirchhoff equations.



The Kirchhoff equation goes back to Kirchhoff in 1883.
It was proposed as an extension of the classical D'Alembert's wave equation for the vibration of elastic strings. The equation is one dimensional, time dependent, and it was written as

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \ ,$$

where L is the length of the string, h is the area of cross-section, E is the young modulus (elastic modulus) of the material, ρ is the mass density, and P_0 is the initial tension. Almost one century later, Jacques Louis Lions returned to the equation and proposed an abstract framework for the general Kirchhoff equation in higher dimension with external force term. Lions equation was written as

$$\frac{\partial^2 u}{\partial t^2} + \left(a + b \int_\Omega |\nabla u|^2 dx \right) \Delta u = f(x, u)$$

where $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ is the Laplace-Beltrami (Euclidean) Laplacian.

Let (M^n, g) be a closed Riemannian n-manifold, $n \geq 3$. Let $a, b, \theta_0 > 0$ be positive real numbers, and $h : M \to \mathbb{R}$ be a C^1-function in M. The Kirchhoff equation with power that we investigate (following a question by M. Struwe) is written as

$$\left(a + b\int_M |\nabla u|^2 dv_g\right)^{\theta_0} \Delta_g u + hu = u^{p-1} \ , \qquad (KE)$$

where $\Delta_g = -\mathrm{div}_g \nabla$ is the Laplace-Beltrami operator, $2 < p \leq 2^\star$, $2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent, and we require that $u \geq 0$ in M. As usual : the equation is subcritical if $p < 2^\star$ and critical if $p = 2^\star$.

Goal : investigate existence and compactness of the equation.

Note : compactness is investigated through the Palais-Smale property and/or the notion of stability.

3. The notion of stability.

Stability is associated to the notion of perturbations of (KE).

Definition (of a perturbation of (KE))

A perturbation of (KE) is any family $(KE_\alpha)_\alpha$ of equations which are written as

$$\left(a_\alpha + b_\alpha \int_M |\nabla u_\alpha|^2 dv_g\right)^{\theta_0} \Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{p_\alpha - 1} , \qquad (KE_\alpha)$$

where $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ are arbitrary sequences of positive real numbers converging to a and b, $(h_\alpha)_\alpha$ is an arbitrary sequence of C^1-functions converging C^1 to h and $(p_\alpha)_\alpha$ is an arbitrary sequence of powers converging to p in $(2, 2^\star]$.

An obvious remark is that (KE) is a perturbation of itself (take $a_\alpha = a$, $b_\alpha = b$, $h_\alpha = h$ and $p_\alpha = p$ for all α).

Definition (Bounded Stability)

Equation (KE) is said to be bounded and stable (or stable for short) if for any sequences $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ of positive real numbers converging to a and b, any sequence $(h_\alpha)_\alpha$ of C^1-functions converging C^1 to h, any sequence $(p_\alpha)_\alpha$ of powers converging to p in $(2, 2^\star]$, and any sequence $(u_\alpha)_\alpha$ of nonnegative solutions of (KE_α) there holds that, up to passing to a subsequence, $(u_\alpha)_\alpha$ converges strongly in C^2 to a nonnegative solution of (KE).

Bounded stability means that whatever the perturbations (KE_α) of our equation (KE) we consider, whatever the sequence $(u_\alpha)_\alpha$ of solutions of (KE_α) we consider, the sequence $(u_\alpha)_\alpha$ converges strongly (up to a subsequence) in C^2 (to a solution of our original equation).

Since (KE) is a perturbation of itself, stability $\Rightarrow$ compactness .

3. The subcritical case.

The kind of typical results we aim to prove is the following.

Theorem 1 (Subcritical case, H. 2016)

Let $p \in (2, 2^\star)$. Assume $\Delta_g + \frac{h}{a^{\theta_0}}$ is coercive and $2(1+\theta_0) \neq p$. Then :

(1) [Existence] (KE) always possesses a C^2-positive solution.

(2) [Stability] For any perturbation $(KE_\alpha)_\alpha$ of (KE) and any sequence $(u_\alpha)_\alpha$ of solutions of $(KE_\alpha)_\alpha$ there holds that, up to passing to a subsequence, $(u_\alpha)_\alpha$ converges strongly in C^2 to a nonnegative solution of (KE).

As a direct consequence of the stability part in the theorem (and the previous remark), assuming that $\Delta_g + \frac{h}{a^{\theta_0}}$ is coercive :

Corollary

(KE) is compact for any $p \in (2, 2^\star)$ such that $2(1+\theta_0) \neq p$.

Proof of Theorem 1 : (1) [Existence] The problem has a variational structure. The proof follows from the standard Brézis and Nirenberg mountain pass approach, based on the mountain-pass lemma by Ambrosetti and Rabinowitz. The primitive functional for (KE) is

$$I_p(u) = \frac{1}{2(1+\theta_0)b}\left(a + b\int_M |\nabla u|^2 dv_g\right)^{1+\theta_0} + \frac{1}{2}\int_M hu^2 dv_g - \frac{1}{p}\int_M (u^+)^p dv_g ,$$

where $u \in H^1$, and H^1 is the Sobolev space of functions in L^2 with one derivative in L^2. Since M has no boundary, constant functions can be used to go down the mountain, and we recover a mountain pass structure. Subcriticality gives the required compactness to have the argument work.

Proof of Theorem 1 continued : (2) [Stability] We prove it in 3 (by now) simple steps. We let $(u_\alpha)_\alpha$ be arbitrary as in the theorem, solution of $(KE_\alpha)_\alpha$. We define

$$K_\alpha = \left(a_\alpha + b_\alpha \int_M |\nabla u_\alpha|^2 dv_g \right)^{\theta_0} ,$$

$$H_\alpha = \frac{h_\alpha}{K_\alpha} , \ v_\alpha = K_\alpha^{-\frac{1}{p_\alpha - 2}} u_\alpha .$$

There holds that

$$\Delta_g v_\alpha + H_\alpha v_\alpha = v_\alpha^{p_\alpha - 1} .$$

We branch on the Gidas-Spruck compactness argument. We prove that :

Step 1 : (easy) We cannot have simultaneously that $\|\nabla u_\alpha\|_{L^2} \to +\infty$ as $\alpha \to +\infty$ and $\|v_\alpha\|_{L^\infty} = O(1)$.

Step 2 : (very easy) $(u_\alpha)_\alpha$ converges in C^2 if $\|v_\alpha\|_{L^\infty} = O(1)$.

Step 3 : (the Gidas-Spruck argument) We cannot have that $\|v_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$.

4. The critical dimension in the critical case $p = 2^\star$.

A critical dimension appears in the critical case of our equations. It acts as a threshold in dimensions in our results.

Definition (Critical dimension, H.2016)

We let $d_c > 0$ be given by

$$d_c = \frac{2(1+\theta_0)}{\theta_0} ,$$

and we refer to d_c as the fractional critical dimension of (KE) when $p = 2^\star$.

It might be that $d_c \notin \mathbb{N}$. There holds that $d_c \in \mathbb{N}$ iff $\theta_0 = 2, 1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \ldots$ (i.e. $\theta_0 = \frac{2}{m-2}$ for $m \in \mathbb{N}$, $m \geq 3$.)

In what follows (KE_c) refers to (KE) with $p = 2^\star$.

5. The H^1-theory and the Palais-Smale property.

Let $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ be sequences of positive real numbers converging to a and b, $(h_\alpha)_\alpha$ be a sequence in C^1 converging C^1 to h and $(p_\alpha)_\alpha$ be a sequence converging to $2^\star$ in $(2, 2^\star]$. For $u \in H^1$, and $\alpha \in \mathbb{N}$, we let

$$
\begin{aligned}
I_\alpha(u) = &\frac{1}{2(1+\theta_0)b_\alpha}\left(a_\alpha + b_\alpha \int_M |\nabla u|^2 dv_g\right)^{1+\theta_0} \\
&+ \frac{1}{2}\int_M h_\alpha u^2 dv_g - \frac{1}{p_\alpha}\int_M |u|^{p_\alpha} dv_g \ .
\end{aligned}
$$

Nonnegative critical points of I_α are solutions of (KE_α).

A sequence $(u_\alpha)_\alpha$ in H^1 is said to be a Palais-Smale sequence for the functional family $(I_\alpha)_\alpha$ if $(I_\alpha(u_\alpha))_\alpha$ is bounded and $I'_\alpha(u_\alpha) \to 0$ in $(H^1)'$ as $\alpha \to +\infty$.

The functional family $(I_\alpha)_\alpha$ is said to satisfy the Palais-Smale property if for any Palais-Smale sequence $(u_\alpha)_\alpha$ for $(I_\alpha)_\alpha$ there holds that, up to a subsequence, $(u_\alpha)_\alpha$ converges (strongly) in H^1.

Background Theorem 1 (Struwe's H^1-theory)

Let $(u_\alpha)_\alpha$ be a H^1-bounded Palais-Smale sequence for $(I_\alpha)_\alpha$. Up to a subsequence, either $(u_\alpha)_\alpha$ converges strongly in H^1 or

$$u_\alpha = u_\infty + \sum_{i=1}^{k} c_i K_\alpha^{1/(p_\alpha-2)} \mathcal{B}_\alpha^i + \mathcal{R}_\alpha$$

in M for some $k \in \mathbb{N}^\star$, where $u_\infty : M \to \mathbb{R}$ is the weak limit in H^1 (or the strong limit in L^2) of the u_α's, K_α is given by

$$K_\alpha = \left(a_\alpha + b_\alpha \int_M |\nabla u_\alpha|^2 dv_g \right)^{\theta_0} ,$$

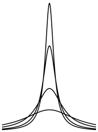
$\mathcal{R}_\alpha \to 0$ in H^1 as $\alpha \to +\infty$, the $(\mathcal{B}_\alpha^i)_\alpha$'s are bubbles, and the c_i's are real numbers in $[1, +\infty)$ which all equal 1 in the purely critical case for which $p_\alpha = 2^\star$ for all α.

In the arbitrary sign case, bubbles are constructed by rescaling solutions in $\mathbb{R}^n$ of the fundamental equation $\Delta u = |u|^{2^\star-2}u$. In the nonnegative case, thanks to Caffarelli-Gidas-Spruck, an explicit expression for the $\mathcal{B}_\alpha^i$'s can be given and there holds that

$$\mathcal{B}_\alpha^i(x) = \left(\frac{\mu_{i,\alpha}}{\mu_{i,\alpha}^2 + \frac{d_g(x_{i,\alpha},x)^2}{n(n-2)}}\right)^{\frac{n-2}{2}}$$

for all $x \in M$ and all α, where $(x_{i,\alpha})_\alpha$ is a converging sequence of points in M and $(\mu_{i,\alpha})_\alpha$ is a sequence of positive real numbers converging to 0 as $\alpha \to +\infty$. Bubbles $(\mathcal{B}_\alpha)_\alpha$ satisfy that

$$\mathcal{B}_\alpha \to 0 \text{ in } L^2 \text{ as } \alpha \to +\infty\ ,$$
$$\|\mathcal{B}_\alpha\|_{H^1} = O(1)\ , \text{ and,}$$
$$\|\mathcal{B}_\alpha\|_{H^1}^2 \geq S^{n/2} + o(1) \text{ for all } \alpha\ ,$$



where S is the sharp constant in the Sobolev inequality. In the case of positive bubbles, equality holds in the last equation.

Boundedness Lemma 1 (H. 2016)

Whatever $(I_\alpha)_\alpha$ is,

(i) Palais-Smale sequences of nonnegative functions for $(I_\alpha)_\alpha$ are bounded in H^1 either when $n \neq d_c$, or when $n = d_c$ and $bS^{(1+\theta_0)/\theta_0} > 1$,

(ii) Palais-Smale sequences of functions of arbitrary sign for $(I_\alpha)_\alpha$ are bounded in H^1 if $\Delta_g + \frac{h}{a^{\theta_0}}$ is coercive and $n \neq d_c$,

(iii) In the purely critical case, where $p_\alpha = 2^\star$ for all α, Palais-Smale sequences of functions of arbitrary sign for $(I_\alpha)_\alpha$ are bounded in H^1 if $\Delta_g + \frac{h}{a^{\theta_0}}$ is coercive, $n = d_c$ and $b \gg 1$,

where d_c the critical fractional dimension and S is the sharp Sobolev constant.

$\Rightarrow$ If $\theta_0 \notin \left\{2, 1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \dots\right\}$ and $\Delta_g + \frac{h}{a^{\theta_0}}$ is coercive, then Palais-Smale sequences for $(I_\alpha)_\alpha$ are bounded in H^1.

Proof of Lemma 1 : Lemma 1 follows from relatively easy and standard manipulations starting from the equations (somehow in the spirit of the Brézis-Nirenberg manipulations), using the Poincaré and Sobolev inequalities. $\diamondsuit$

Theorem 2 (Bubble control 1, H. 2016)

Let $(I_\alpha)_\alpha$ be an arbitrary family of functionals as above. Let $(u_\alpha)_\alpha$ be a H^1-bounded Palais-Smale sequences for $(I_\alpha)_\alpha$. Suppose $(u_\alpha)_\alpha$ blows-up with k bubbles. Then $bkS^{n/2} < 1$ if $n = d_c$ and

$$a^{\kappa_0 - 1} bk \le \frac{(\kappa_0 - 1)^{\kappa_0 - 1}}{\kappa_0^{\kappa_0} S^{n/2}} \qquad (\star)$$

if $n > d_c$, where d_c is the critical fractional dimension, S is the sharp Sobolev constant and $\kappa_0 = \frac{2\theta_0}{2^\star - 2}$.

Proof of Theorem 2 : This is still a theorem we can prove with (relatively) easy arguments. Key point is the energy estimate associated to Struwe's decomposition (background Theorem 1). $\diamondsuit$

Corollary

If $\Delta_g + \frac{h}{a^{\theta_0}}$ is coercive, $n > d_c$ and

$$a^{\kappa_0 - 1} b > \frac{(\kappa_0 - 1)^{\kappa_0 - 1}}{\kappa_0^{\kappa_0} S^{n/2}} ,$$

then any family of functionals $(I_\alpha)_\alpha$ as above satisfies the Palais-Smale property.

Proof of the Corollary : By Lemma 1, since $\Delta_g + \frac{h}{a^{\theta_0}}$ is coercive and $n \neq d_c$, Palais-Smale sequences for $(I_\alpha)_\alpha$ are H^1-bounded. Then, up to passing to a subsequence, any Palais-Smale sequence can be decomposed using Struwe's decomposition (background Theorem 1). Assuming the inequality in the corollary, and since $n > d_c$, it follows from Theorem 2 that $k = 0$ (zero bubble case) in these decompositions. In other words, up to passing to a subsequence, Palais-Smale sequences for $(I_\alpha)_\alpha$ converge strongly in H^1. This ends the proof of the corollary. $\diamondsuit$

6. The bounded stability approach.

Here again we need a boundedness (finite energy) lemma.

Boundedness Lemma 2 (H. 2016)

For any perturbation (KE_α) *of* (KE_c)*, and any sequence* $(u_\alpha)_\alpha$ *of nonnegative solutions of* (KE_α)*, the sequence* $(u_\alpha)_\alpha$ *is bounded in* H^1 *in each of the following cases :*

- *(i)* $n = 3$ *and* $\theta_0 < 2$,
- *(ii)* $n > d_c$,
- *(iii)* $n = d_c$ *and* $bS^{n/2} > 1$,
- *(iv)* $n \geq 4$ *and* $S_g > 0$,

where d_c *is the critical fractional dimension,* S *is the sharp Sobolev constant and* S_g *is the scalar curvature of* g.

Proof of Lemma 2 : (ii) and (iii) are proved as in Lemma 1. We prove (i) and (iv) with (much) more advanced arguments based on the C^0-theory and the bounded stability theory for the blow-up of nonlinear critical elliptic equations (both in the $n = 3$ and $n \geq 4$ cases). $\diamondsuit$

As a remark, when (i), (ii), (iii) or (iv) apply, then any sequence $(u_\alpha)_\alpha$ of nonnegative solutions of (KE_α) is a Palais-Smale sequence for $(I_\alpha)_\alpha$, and what has been said in the previous section applies.

We can think of bounded stability as sort of a Palais-Smale property but restricted to sequences of solutions of equations like (KE_α). Of course one expects to get more precise results.

As in the Palais-Smale case we can ask for a bubble control in case of blow-up. We let k_{max} be the maximal number of bubbles that blowing-up sequences of solutions of perturbed equations (KE_α) associated to a given equation (KE_c) can have. Possibly $k_{max} = +\infty$, in particular if there is one sequence (KE_α) which possesses a unbounded sequence in H^1 of solutions.

According to the above remarks, and to Theorem 2, we know that if $n = d_c$, and either $S_g > 0$ (and $n \geq 4$) or $bS^{n/2} > 1$, then

$$bk_{max}S^{n/2} \leq 1 \ . \qquad (H1Ineq1)$$

And we also know that if $n > d_c$, then

$$a^{\kappa_0 - 1}bk_{max} \leq \frac{(\kappa_0 - 1)^{\kappa_0 - 1}}{\kappa_0^{\kappa_0} S^{n/2}} \ , \qquad (H1Ineq2)$$

where $\kappa_0 = \frac{2\theta_0}{2^\star - 2}$. Indeed, in these cases, any sequence $(u_\alpha)_\alpha$ of solutions of (KE_α) is bounded in H^1 by Lemma 2. Then it is a Palais-Smale sequence for $(I_\alpha)_\alpha$. In particular the bubble control theorem of the preceding section applies. And we get the above two equations.

More can be said. This is the subject of the following theorem. We recall that by the positive mass theorem (Schoen-Yau, Witten) there always exists a positive function Λ_g in M such that $\Delta_g + \Lambda_g$ has positive mass on any 3-manifold of positive scalar curvature.

Theorem 3 (Bubble control 2, H. 2016)

When $n = 3$ we assume that $\theta_0 < 2$ and that $\Delta_g + \frac{h}{a^{\theta_0}}$ is coercive. We let d_c be the critical fractional dimension. We assume that $S_g > 0$ in M. If $n \geq d_c$, there holds that

$$k_{max} \leq \frac{\left(C^{1/\theta_0} - a\right)^+}{bS^{n/2}a^{\kappa_0}} , \tag{1}$$

where $C > 0$ is such that $h \leq C\Lambda_g$ in M and $\Lambda_g = \frac{n-2}{4(n-1)}S_g$. If $n < d_c$, there holds that

$$k_{max} \leq \frac{\left(C^{1/\theta_0} - a\right)^+}{bS^{n/2}C^{\kappa_0/\theta_0}} , \tag{2}$$

where $C > 0$ is such that $h \leq C\Lambda_g$ in M and $\Lambda_g > 0$ is such that $\Delta_g + \Lambda_g$ has positive mass when $n = 3$, or $\Lambda_g = \frac{n-2}{4(n-1)}S_g$ when $n \geq 4$.

Together with Theorem 3 we do get stability result by assuming inequalities which imply that $k_{max} = 0$. Consider the three following inequalities :

$$\text{(Ineq.1)} \quad a^{\kappa_0 - 1} b > \frac{(\kappa_0 - 1)^{\kappa_0 - 1}}{\kappa_0^{\kappa_0} S^{n/2}} \ ,$$

$$\text{(Ineq.2)} \quad h < \frac{(n-2)a^{\theta_0}}{4(n-1)} \left(1 + bS^{n/2} a^{\kappa_0 - 1}\right)^{\theta_0} S_g \ ,$$

$$\text{(Ineq.3)} \quad h < \left(a + \left(bS^{n/2}\kappa_0\right)^{\frac{1}{1-\kappa_0}}\right)^{\theta_0} \Lambda_g \ ,$$

where $\Lambda_g > 0$ is such that $\Delta_g + \Lambda_g$ has positive mass when $n = 3$, or $\Lambda_g = \frac{n-2}{4(n-1)} S_g$ when $n \geq 4$, and where $\kappa_0 = \frac{2\theta_0}{2^\star - 2}$.

(Ineq.2) and (Ineq.3) are like

$$h \leq C(n, a, b, \theta_0) \Lambda_g$$

(Aubin's type inequalities), where $\Lambda_g > 0$ is such that $\Delta_g + \Lambda_g$ has positive mass when $n = 3$ and $\Lambda_g = \frac{n-2}{4(n-1)} S_g$ when $n \geq 4$.

Corollary 1 (Bounded stability, H. 2016)

Let (M^n, g) be a closed Riemannian n-manifold, $a, b, \theta_0 > 0$ be positive real numbers, and $h : M \to \mathbb{R}$ be a C^1-function in M. When $n = 3$ we assume that $\theta_0 < 2$ and that $\Delta_g + \frac{h}{a^{\theta_0}}$ is coercive. Equation (KE_c) is bounded and stable in each of the following cases : if $n = d_c$ and $bS^{n/2} > 1$, or if $n > d_c$ and (Ineq.1) holds true, or if $S_g > 0$, $n \geq d_c$ and (Ineq.2) holds true, or if $S_g > 0$, $n < d_c$ and (Ineq.3) holds true.

Another corollary to the theorem concerns existence.

Corollary 2 (Existence, H. 2016)

Let (M^n, g) be a closed Riemannian n-manifold, $a, b, \theta_0 > 0$ be positive real numbers, and $h : M \to \mathbb{R}$ be a C^1-function in M. We assume that $\theta_0 < 2$ when $n = 3$, and also assume in all dimensions that $\Delta_g + \frac{h}{a^{\theta_0}}$ is coercive. Then (KE_c) possesses a C^2-positive solution in each of the cases listed in Corollary 1.

Proof of Corollary 1 : (Ineq.1), (Ineq.2) and (Ineq.3) imply that $k_{max} = 0$, and thus that for any perturbed sequence of equation (KE_α) and any sequence $(u_\alpha)_\alpha$ of solutions of (KE_α), the u_α's do not blow-up. Thus, up to a subsequence, they converge strongly in H^1, and by regularity theory they converge in C^2. $\diamondsuit$

Proof of Corollary 2 : existence follows from the existence of a solution in the subcritical case (Theorem 1) together with the stability from Theorem 3 (Corollary 1 above). The subcritical solution converges to a solution of the critical equation. $\diamondsuit$

Theorem 4 (A special case, H.2016)

Let (M^n, g) be a closed Riemannian n-manifold of positive scalar curvature and dimension $n = 4, 5$. Let $a, b, \theta_0 > 0$ be positive real numbers. Let $(u_\alpha)_\alpha$ be a blowing-up sequence of solutions of the Kirchhoff equation (KE_c) with $h \equiv \frac{n-2}{4(n-1)} S_g$. Let k be the number of bubble involved in the Struwe's decomposition of the u_α's. Then $\frac{1-a}{b} = kS^{n/2}$. In particular, (KE) is compact if $\frac{1-a}{b} \notin \mathbb{N}^\star S^{n/2}$.

7. The opposite question. Blowing-up examples.

On the opposite side we want to produce examples of solutions which blow up. The following result is a direct consequence of previous works by Pistoia and Vétois.

Proposition (Pistoia-Vétois)

Let (M^n, g) be a closed Riemannian n-manifold of nonpositive scalar curvature. Assume that $n = d_c$ and $\theta_0 \neq 2$, where d_c is the critical fractional dimension. Then there exist $h : M \to \mathbb{R}^{+\star}$ a positive C^1-function, $a, b > 0$ with $bS^{n/2} = 1$, sequences $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ of positive real numbers converging to a and b, a sequence $(h_\alpha)_\alpha$ of positive C^1-function converging to h, a sequence $(p_\alpha)_\alpha$ of powers converging to $2^\star$ in $[2, 2^\star]$ and a sequence $(u_\alpha)_\alpha$ of solutions of (KE_α) such that $\int_M |\nabla u_\alpha|^2 dv_g \to +\infty$ as $\alpha \to +\infty$, namely such that $(u_\alpha)_\alpha$ is not bounded in H^1.

Such constructions are not possible when $S_g > 0$ (and $n \geq 4$) or when $n > d_c$. In the same spirit : Del Pino, Esposito, Ghimenti, Micheletti, Pacard, Pistoia, Robert, Vaira, Vétois, Wei.

8. Proof of Theorem 3.

The proof is based on the energy estimate associated to the background theorem we discussed a few slides ago, and on the following (difficult blow-up) results from the stability theory for nonlinear critical elliptic equations. Consider a sequence $(v_\alpha)_\alpha$ of solutions of equations like

$$\Delta_g v_\alpha + H_\alpha v_\alpha = v_\alpha^{p_\alpha - 1} ,$$

where $H_\alpha \to H_\infty$ in C^1 and $p_\alpha \to 2^\star$ in $(2, 2^\star]$. Then,

(BR1) (Li-Zhu, Schoen) If $n = 3$ and $\Delta_g + H_\infty$ has positive mass, then, up to passing to a subsequence, the v_α's converge in C^2,

(BR2) (Druet, Druet-H.) If $n \geq 4$ and $H_\infty < \frac{n-2}{4(n-1)} S_g$ in M, then, up to passing to a subsequence, the v_α's converge in C^2.

Now we come back to energy arguments. We let $(u_\alpha)_\alpha$ be a blowing-up sequence of solutions of equations like (KE_α). We assume that $S_g > 0$. By the boundedness lemma 2, $(u_\alpha)_\alpha$ is bounded in H^1 and thus is a Palais-Smale sequence for $(I_\alpha)_\alpha$. It follows from the background theorem we stated a few slides above that

$$K_\alpha^{1/\theta_0} = a_\alpha + b_\alpha \int_M |\nabla u_\infty|^2 dv_g + b_\alpha \sum_{i=1}^k c_i^2 K_\alpha^{2/(p_\alpha - 2)} S_i^{n/2} + o(1) \ ,$$

where $S_i^{n/2} = \lim_{\alpha \to +\infty} \|\nabla \mathcal{B}_\alpha^i\|_{L^2}^2$. Thus

$$K_\infty^{1/\theta_0} = a + b \int_M |\nabla u_\infty|^2 dv_g + b K_\infty^{2/(2^\star - 2)} C_0 \ ,$$

where $C_0 \geq k S^{n/2}$. Let $X_0 = K_\infty^{1/\theta_0}$. Then

$$X_0 = a + b \int_M |\nabla u_\infty|^2 dv_g + b C_0 X_0^{\kappa_0} \ .$$

We define $f : \mathbb{R}^+ \to \mathbb{R}$ by

$$f(X) = bkS^{n/2} X^{\kappa_0} - X + a \ .$$

By the above equation, since $C_0 \geq k S^{n/2}$, there holds that $f(X_0) \leq 0$. First we assume that $n \geq d_c$. This assumption together with the assumption that $\theta_0 < 2$ if $n = 3$, implies that $n \geq 4$.

We define

$$H_\alpha = \frac{h_\alpha}{K_\alpha} \quad \text{and} \quad v_\alpha = K_\alpha^{-\frac{1}{p_\alpha - 2}} u_\alpha \ ,$$

where K_α is as before. Then,

$$\Delta_g v_\alpha + H_\alpha v_\alpha = v_\alpha^{p_\alpha - 1} \ .$$

By (BR2) above, since the v_α's blow up, there holds that there need to exist $x_0 \in M$ such that

$$\frac{h(x_0)}{K_\infty} \geq \frac{n-2}{4(n-1)} S_g(x_0) \ .$$

Then,

$$X_\star \leq X_0 \leq C^{1/\theta_0} \ ,$$

where $X_\star$ is the smallest $X \geq 0$ such that $f(X) \leq 0$ and $C > 0$ is as in the theorem. If $n = d_c$, then $bkS^{n/2} < 1$ and $\kappa_0 = 1$, and we get that $f'(x) < 0$ for all x. In particular, $f'(a) < 0$. If $n > d_c$, then by the inequality of Theorem 2,

$$
\begin{aligned}
f'(a) &= bkS^{n/2}a^{\kappa_0-1}\kappa_0 - 1\\
&\le \left(\frac{\kappa_0-1}{\kappa_0}\right)^{\kappa_0-1} - 1\\
&< 0\ .
\end{aligned}
$$

There also holds that $f(t) > 0$ for $t \in [0, a]$. Obviously f is convex in $\mathbb{R}^+$. Then

$$f(X_\star) \ge f(a) + f'(a)\,(X_\star - a)$$

and since $f\,(X_\star) = 0$, we get that

$$a + \frac{f(a)}{-f'(a)} \le X_\star\ .$$

Noting that $0 < -f'(a) < 1$, and since (see above) $X_\star \le C^{1/\theta_0}$,

$$a + bkS^{n/2}a^{\kappa_0} \le C^{1/\theta_0}\ .$$

This settles the $n \ge d_c$ case.

Now we assume that $n < d_c$. Then $\kappa_0 < 1$. Let $p_0 = 1/\kappa_0$ and $g : \mathbb{R}^+ \to \mathbb{R}$ be the function given by

$$g(X) = X^{p_0} - bkS^{n/2}X - a\ .$$

There holds that $g\left(K_\infty^{\kappa_0/\theta_0}\right) \geq 0$, while g is decreasing up to X_2 and increasing after X_2, where

$$X_2 = \left(\frac{bkS^{n/2}}{p_0}\right)^{1/(p_0-1)}\ .$$

In particular, since $g(0) < 0$, $X_2 \leq K_\infty^{\kappa_0/\theta_0}$. Obviously, $\Delta_g + \frac{h}{K_\infty}$ is coercive if $\Delta_g + \frac{h}{a^{\theta_0}}$ is coercive. Indeed, $K_\infty \geq a^{\theta_0}$, and thus $\frac{1}{K_\infty} = \frac{\varepsilon_0}{a^{\theta_0}}$, where $0 < \varepsilon_0 \leq 1$. Then,

$$\Delta_g + \frac{h}{K_\infty} = \varepsilon_0\left(\Delta_g + \frac{h}{a^{\theta_0}}\right) + (1-\varepsilon_0)\Delta_g$$

and the coercivity of $\Delta_g + \frac{h}{K_\infty}$ follows from the coercivity of $\Delta_g + \frac{h}{a^{\theta_0}}$. By (BR1) and (BR2) above, there need to be one

$x_0 \in M$ s.t. $h(x_0) \geq K_\infty \Lambda_g(x_0)$. Then $K_\infty \leq C$, where $C > 0$ is as in the theorem. In particular, $g\left(C^{\kappa_0/\theta_0}\right) \geq 0$. Thus

$$C^{1/\theta_0} - a \geq bkS^{n/2} C^{\kappa_0/\theta_0} \ ,$$

and this settles the $n < d_c$ case. Theorem 3 is proved. $\diamondsuit$

9. Proof of Theorem 4.

The proof is based on the following (difficult blow-up) results from the stability theory for nonlinear critical elliptic equations : consider a sequence $(v_\alpha)_\alpha$ of solutions of equations like

$$\Delta_g v_\alpha + H_\alpha v_\alpha = v_\alpha^{2^\star - 1} ,$$

where $H_\alpha \to H_\infty$ in C^1. Suppose the v_α's are bounded in H^1. Then,

(BR3) (Druet, Druet-H., Druet-H.-Robert) If $n \neq 6$ and $H_\infty \neq \frac{n-2}{4(n-1)} S_g$ at all points in M, then, up to passing to a subsequence, the v_α's converge in C^2,

(BR4) (Druet) If $n = 4, 5$ and $H_\infty \neq \frac{n-2}{4(n-1)} S_g$ at all points in M, then the sole possible weak limit in H^1 of the v_α is the zero function.

Now we can proceed as follows. We let $(u_\alpha)_\alpha$ be a blowing-up sequence of solutions of our equation. We define

$$H_\alpha = \frac{(n-2)}{4(n-1)K_\alpha} S_g \quad \text{and} \quad v_\alpha = K_\alpha^{-\frac{1}{2^\star-2}} u_\alpha \ ,$$

where $K_\alpha = \left(a + b\int_M |\nabla u_\alpha|^2 dv_g\right)^{\theta_0}$. Then,

$$\Delta_g v_\alpha + H_\alpha v_\alpha = v_\alpha^{2^\star-1} \ .$$

Since $S_g > 0$ and $n \geq 4$ the u_α's are bounded in H^1. By energy arguments, noting that $c_i = 1$ since we are blocked on the critical equation, and that $S_i = S$ since we are talking about positive solutions,

$$K_\alpha^{1/\theta_0} = a + b\int_M |\nabla u_\infty|^2 dv_g + b\sum_{i=1}^k K_\alpha^{2/(2^\star-2)} S^{n/2} + o(1) \ .$$

By (BR3) and (BR4) above there holds that $u_\infty \equiv 0$ and we must have that $K_\infty = 1$. Then, by passing to the limit,

$$1 = a + bkS^{n/2} \ ,$$

and this proves the theorem. $\diamondsuit$

Thank you for your attention !