Klein-Gordon-Maxwell-Proca systems in closed manifolds

by

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I. The full KGMP-system.

Nonlinear Klein-Gordon total functional, minimum coupling rule

$$\partial_t \to \partial_t + iq\varphi \text{ and } \nabla \to \nabla - iqA,$$

where (φ, A) gauge potential representing the electromagnetic field, governed by the Maxwell-Proca Lagrangian. Consider the two Lagrangian densities

$$\mathcal{L}_{NKG}(\psi, \varphi, A)$$
$$= \frac{1}{2}\left|\left(\frac{\partial}{\partial t} + iq\varphi\right)\psi\right|^2 - \frac{1}{2}\left|(\nabla - iqA)\psi\right|^2 - \frac{m_0^2}{2}|\psi|^2 + \frac{1}{p}|\psi|^p$$

and

$$\mathcal{L}_{MP}(\varphi, A) = \frac{1}{2}\left|\frac{\partial A}{\partial t} + \nabla\varphi\right|^2 - \frac{1}{2}|\nabla \times A|^2 + \frac{m_1^2}{2}|\varphi|^2 - \frac{m_1^2}{2}|A|^2 .$$

Here $\nabla \times = \star d$, $\star$ Hodge dual, d differentiation. Massive version of KGM theory. Here ψ matter field, m_0 its mass, q its charge, (A, φ) gauge potentials representing the electromagnetic vector field, m_1 is the Proca mass.

Consider the total action functional

$$\mathcal{S}(\psi, \varphi, A) = \int\int \left(\mathcal{L}_{NKG} + \mathcal{L}_{MP}\right) dv_g dt .$$

Write ψ in polar form as $\psi(x,t) = u(x,t)e^{iS(x,t)}$ for $u \geq 0$ and $u, S : M \times \mathbb{R} \to \mathbb{R}$. Then the total action rewrites as

$$\mathcal{S}(u, S, \varphi, A) = \frac{1}{2}\int\int \left(\left(\frac{\partial u}{\partial t}\right)^2 - |\nabla u|^2 - m_0^2 u^2\right) dv_g dt$$

$$+ \frac{1}{p}\int\int u^p dv_g dt$$

$$+ \frac{1}{2}\int\int \left(\left(\frac{\partial S}{\partial t} + q\varphi\right)^2 - |\nabla S - qA|^2\right) u^2 dv_g dt$$

$$+ \frac{1}{2}\int\int \left(\left|\frac{\partial A}{\partial t} + \nabla\varphi\right|^2 - |\nabla \times A|^2 + \frac{m_1^2}{2}|\varphi|^2 - \frac{m_1^2}{2}|A|^2\right) dv_g dt .$$

We can take the variation of $\mathcal{S}$ with respect to u, S, φ, and A. For instance, if we let ω_g be the volume form of (M,g), then

$$\frac{1}{2}\left(\frac{d}{dA}\int|\nabla\times A|^2\right).(B)=\int(\star dA,\star dB)\,\omega_g \qquad (\text{quadratic} + \nabla\times=\star d)$$

$$=(-1)^{n-1}\int(\star dA,(\star d\star)\star B)\,\omega_g \qquad (\star\star=(-1)^{n-1}\text{ in }\Lambda^1)$$

$$=\int(\star dA,\delta\star B)\,\omega_g \qquad (\delta=(-1)^{n-1}\star d\star\text{ in }\Lambda^{n-1})$$

$$=\int(d\star dA,\star B)\,\omega_g \qquad (\text{Stokes formula})$$

$$=\int(\star\delta dA,\star B)\,\omega_g \qquad (d\star=\star\delta\text{ in }\Lambda^2)$$

$$=\int(\star\delta dA)\wedge(\star\star B) \qquad (\text{since }\alpha\wedge(\star\beta)=(\alpha,\beta)\omega_g\text{ in }\Lambda^p)$$

$$=(-1)^{n-1}\int(\star\delta dA)\wedge B \qquad (\star\star=(-1)^{n-1}\text{ in }\Lambda^1)$$

$$=\int(\delta dA,B)\,\omega_g \qquad (\alpha\wedge\beta=(-1)^{n-1}\beta\wedge\alpha\text{ for }\alpha\in\Lambda^{n-1},\beta\in\Lambda^1)$$

In particular,

$$\frac{1}{2}\left(\frac{d}{dA}\int|\nabla\times A|^2\right).(B)=\int\left(\overline{\Delta}_g A,B\right)$$

for all B, where $\overline{\Delta}_g=\delta d$, δ codifferential. When $n=3$, $\overline{\Delta}_g=\nabla\times\nabla\times$. Taking the variation of

$$\mathcal{S}(\psi,\varphi,A)=\int\int\left(\mathcal{L}_{NKG}+\mathcal{L}_{MP}\right)dv_g dt\ .$$

with respect to u, S, φ, and A, we then get four equations which are written as

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}+\Delta_g u+m_0^2 u=u^{p-1}+\left(\left(\frac{\partial S}{\partial t}+q\varphi\right)^2-|\nabla S-qA|^2\right)u\\ \frac{\partial}{\partial t}\left(\left(\frac{\partial S}{\partial t}+q\varphi\right)u^2\right)-\nabla.\left(\left(\nabla S-qA\right)u^2\right)=0\\ -\nabla.\left(\frac{\partial A}{\partial t}+\nabla\varphi\right)+m_1^2\varphi+q\left(\frac{\partial S}{\partial t}+q\varphi\right)u^2=0\\ \overline{\Delta}_g A+\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t}+\nabla\varphi\right)+m_1^2 A=q\left(\nabla S-qA\right)u^2\ .\end{cases}\qquad (KGMP)$$

This is the nonlinear Klein-Gordon-Maxwell-Proca system. As $m_1\to 0$ (or letting $m_1=0$), the nonlinear KGMP system reduces to the nonlinear Klein-Gordon-Maxwell system.

II. Why do we refer to Maxwell-Proca ?

Assume $n = 3$. Let the electric field E, the magnetic induction H, the charge density ρ, and the current density J be given by

$$\begin{aligned} E &= -\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) , \\ H &= \nabla \times A , \\ \rho &= -\left(\frac{\partial S}{\partial t} + q\varphi\right) qu^2 , \\ J &= (\nabla S - qA)\, qu^2 . \end{aligned}$$

The two last equations in $(KGMP)$ give rise to the first pair of the Maxwell-Proca equations with respect to a matter distribution whose charge and current density are respectively ρ and J.

We get for free the second pair of the Maxwell-Proca equations.

In other words the two last equations in the $(KGMP)$-system can be rewritten as

$$\begin{aligned} &\nabla.E = \rho - m_1^2\varphi \ , \\ &\nabla \times H - \frac{\partial E}{\partial t} = J - m_1^2 A \ , \\ &\nabla \times E + \frac{\partial H}{\partial t} = 0 \ , \ \nabla.H = 0 \ . \end{aligned}$$

The first equation in the $(KGMP)$-system is the nonlinear Klein-Gordon matter equation. Namely

$$\frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^{p-1} + \frac{\rho^2 - |J|^2}{q^2 u^3} \ .$$

The second equation in the $(KGMP)$-system is the charge continuity equation $\frac{\partial \rho}{\partial t} + \nabla.J = 0$, which is equivalent to the Lorentz condition

$$\nabla.A + \frac{\partial \varphi}{\partial t} = 0 \ .$$

The $(KGMP)$-system is equivalent to this system of 6 equations.

The equivalence between the charge continuity equation and the Lorentz condition involves only basic computations (and uses the condition $m_1 \neq 0$). The Maxwell-Proca equations are written as

$$\nabla.E = \rho - m_1^2\varphi \ , \quad \nabla \times H - \frac{\partial E}{\partial t} = J - m_1^2 A \ ,$$
$$\nabla \times E + \frac{\partial H}{\partial t} = 0 \ , \ \nabla.H = 0 \ .$$

The charge continuity equation states that $\frac{\partial \rho}{\partial t} + \nabla.J = 0$. Taking the derivation of the first Maxwell equation with respect to time, and the divergence of the second equation,

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla.J &= \nabla.\frac{\partial E}{\partial t} + m_1^2\frac{\partial \varphi}{\partial t} + \nabla.(\nabla \times H) - \nabla.\frac{\partial E}{\partial t} + m_1^2\nabla.A \\ &= m_1^2\left(\nabla.A + \frac{\partial \varphi}{\partial t}\right)\end{aligned}$$

since $\nabla.(\nabla \times H) = \delta(\star d)H$, $\delta = \star^{-1}d\star$ in Λ^1, $\star\star = 1$ in Λ^2, and $d^2 = 0$ so that $\nabla.(\nabla \times H) = 0$. The condition $m_1 \neq 0$ breaks the gauge invariance and enforces the Lorentz gauge.

IIBis. A short physics break

The Maxwell equations in Proca form are

$$\begin{aligned} &\nabla . E = \rho - m_1^2 \varphi \ , \\ &\nabla \times H - \frac{\partial E}{\partial t} = J - m_1^2 A \ , \\ &\nabla \times E + \frac{\partial H}{\partial t} = 0 \ , \ \nabla . H = 0 \ . \end{aligned}$$

They reduce to the Maxwell equations as $m_1 \to 0$. Proca (1936) was using the Lorentz formalism. Under this form, referred to as the "modern format", the equations appeared for the first time in a paper by Schrödinger : "*The earth's and the sun's permanent magnetic fields in the unitary field theory*" (1943). These equations have been discussed by several physicists including, in addition to Proca and Schrödinger, people like De Broglie, Pauli , Yukawa, and Stueckleberg. . . The whole point in these theories is that m_1 is nothing but than the mass of the photon : we are talking about a theory where photons have a mass.

Alexandru Proca (1897-1955)

Some possible references :

[1] G.T.Gillies, J.Luo, L.C.Tu, The mass of the photon, Report on Progress in Physics, 68, 2005, 77–130.

[2] A.S.Goldhaber, M.M.Nieto, Photon and Graviton mass limits, Reviews of Modern Physics, 82, 2010, 939–979.

[3] H.Ruegg M. Ruiz-Altaba, The Stueckleberg field, International Journal of Modern Physics A, 19, 2004, 3265–3348.

III. The KGMP-system in reduced form.

Return to

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \Delta_g u + m_0^2 u = u^{p-1} + \left(\left(\frac{\partial S}{\partial t} + q\varphi\right)^2 - |\nabla S - qA|^2\right) u \\ \frac{\partial}{\partial t}\left(\left(\frac{\partial S}{\partial t} + q\varphi\right) u^2\right) - \nabla . \left((\nabla S - qA) u^2\right) = 0 \\ -\nabla . \left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + m_1^2\varphi + q\left(\frac{\partial S}{\partial t} + q\varphi\right) u^2 = 0 \\ \overline{\Delta}_g A + \frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + m_1^2 A = q\left(\nabla S - qA\right) u^2 \ . \end{cases} \qquad (KGMP)$$

Assume A and φ depend on the sole spatial variables (static case), and look for standing waves solutions $u(x)e^{-i\omega t}$. The fourth equation gives that

$$\overline{\Delta}_g A + (q^2 u^2 + m_1^2) A = 0 \ .$$

This implies $A \equiv 0$ since $\int(\overline{\Delta}_g A, A) = \int |dA|^2$. Since $S = -\omega t$ the second equation is automatically satisfied.

The full system reduces to its first and third equation. Letting $\varphi = \omega v$, these first and third equations are written as

$$\begin{cases} \Delta_g u + m_0^2 u = u^{p-1} + \omega^2 (qv - 1)^2 u \\ \Delta_g v + \left(m_1^2 + q^2 u^2\right) v = qu^2 \, . \end{cases} \qquad (S_\omega)$$

Here ω is the phase (or temporal frequency), $m_0, m_1 > 0$ are masses, $q > 0$ is an electric charge, $\Delta_g = -\mathrm{div}_g \nabla$ is the Laplace-Beltrami operator.

When we investigate (S_ω) we talk about standing waves solutions for the full $(KGMP)$-system in static form.

Let $2^\star = \frac{2n}{n-2}$ be the critical Sobolev exponent. With respect to the first equation in (S_ω) the system is energy subcritical when $p < 2^\star$ and energy critical when $p = 2^\star$.

The second equation is subcritical when $n = 3$, critical when $n = 4$, and supercritical when $n \geq 5$.

Some references

The time evolution full system has been investigated in its linear part (without the nonlinear u^{p-1}-term, and $m_1 = 0$) by several people among who Cuccagna, Keel, Klainerman, Machedon, Rodnianski, Roy, Selberg, Sterbenz, Tataru, and Tao (local well-posedness, global well-posedness $n = 3$, global well-posedness for small inital data $n \geq 4$, critical dimension $n = 4$).

The reduced system has been studied in the subcritical case ($p < 2^\star$ and $m_1 = 0$), in dimension 3, by several people among who D'Aprile, d'Avenia, Azzollini, Benci, Bonanno, Cassani, Fortunato, Georgiev, Ghimenti, Mugnai, Pisani, Pompino, Siciliano, Vaira, Visciglia (existence of a solution).

IV. The results.

Let (M, g) be smooth compact of dimension $n \geq 3$, $\partial M = \emptyset$. Let $m_0, m_1 > 0$ and $q > 0$. Let $\omega \in (-m_0, +m_0)$, and $p \in (2, 2^\star]$. We consider the electrostatic KGMP reduced system

$$\begin{cases} \Delta_g u + m_0^2 u = u^{p-1} + \omega^2 (qv - 1)^2 u \\ \Delta_g v + \left(m_1^2 + q^2 u^2\right) v = qu^2 . \end{cases} \qquad (S_\omega)$$

We assume $m_1 > 0$ (Proca formalism). If not the case, $v = \frac{1}{q}$ and the two equations are independent one from another. If $\left(u_\alpha e^{-i\omega_\alpha t}\right)_\alpha$ and $(v_\alpha)_\alpha$ solve our system, then

$$\begin{cases} \Delta_g u_\alpha + m_0^2 u_\alpha = u_\alpha^{p-1} + \omega_\alpha^2 (qv_\alpha - 1)^2 u_\alpha \\ \Delta_g v_\alpha + \left(m_1^2 + q^2 u_\alpha^2\right) v_\alpha = qu_\alpha^2 . \end{cases} \qquad (S_\alpha)$$

The soliton family $\left(u_\alpha e^{-i\omega_\alpha t}\right)_\alpha$ has finite energy if $\|u_\alpha\|_{H^1} = O(1)$. Let $S_p(\omega) = \left\{ \left(ue^{-i\omega t}, v\right),\ u, v > 0 \text{ smooth, which solve } (S_\omega) \right\}$, and for $\mathcal{U} = \left(ue^{-i\omega t}, v\right)$, let also $\|\mathcal{U}\|_{C^{2,\theta}} = \|u\|_{C^{2,\theta}} + \|v\|_{C^{2,\theta}}$.

Definition (a priori bound, stable phase, resonant state)

Let (M,g) be a smooth compact Riemannian manifold of dimensions $n \geq 3$. Let $m_0, m_1 > 0$, $q > 0$, and $p \in (2, 2^\star]$. Let $\omega \in (-m_0, +m_0)$. We say that

(i) ω gives rise to the a priori bound property if there exist $\varepsilon > 0$ and $C > 0$ such that $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$ for all $\mathcal{U} \in S_p(\tilde{\omega})$ and all $\tilde{\omega} \in (\omega - \varepsilon, \omega + \varepsilon)$,

(ii) ω is a stable phase if for any sequence $\left(u_\alpha e^{-i\omega_\alpha t}\right)_\alpha$ of finite energy standing waves, and any sequence $(v_\alpha)_\alpha$ of gauge electric fields, solutions of (S_α), the convergence $\omega_\alpha \to \omega$ in $\mathbb{R}$ implies that, up to a subsequence, $u_\alpha \to u$ and $v_\alpha \to v$ in C^2, where $ue^{-i\omega t}$ and v solve (S_ω).

At last we say that ω is a resonant phase, or give rise to resonant states, if there exist a sequence $\left(u_\alpha e^{-i\omega_\alpha t}\right)_\alpha$ of finite energy standing waves, and a sequence $(v_\alpha)_\alpha$ of gauge electric fields, solutions of (S_α), s.t. $\omega_\alpha \to \omega$ and $\|u_\alpha\|_{L^\infty} + \|v_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$.

A priori bound property $\Rightarrow$ phase stability property

Goals : prove the existence of solutions to our systems, $^{(\star)}$ prove the a priori bound property, prove the phase stability property when the a priori bound property does not hold true, and prove the existence of resonant states when the phase stability property does not hold true.

In the subcritical case :

Theorem 0 (Subcritical case ; Druet-H., 2010 ; H.Truong, 2012)

Let (M,g) be a smooth compact Riemannian n-dimensional manifold, $n \geq 3$, $m_0, m_1 > 0$, and $q > 0$. Let $p \in (2, 2^\star)$. For any $\omega \in (-m_0, +m_0)$ there exists a smooth positive mountain pass solution of (S_ω). Moreover, for any $\theta \in (0,1)$, there exists $C > 0$ such that $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$ for all $\mathcal{U} \in S_p(\omega)$ and all $\omega \in (-m_0, +m_0)$.

N.B. : Thm 0 prevents existence of standing waves with arbitrarily large amplitudes.

$^{(\star)}$: We look for variational solutions such as mountain pass solutions (ground states models in the Nehari-Rabinowitz sense).

Question : when $p = 2^\star$ what should we require on m_0, m_1, and ω in order to get a similar result ? What about resonant states ?

Theorem 1 (A priori bounds ; Druet-H., 2010 ; H.-Truong, 2012)

Let (M, g) be a smooth compact Riemannian manifold of dimensions $n = 3, 4$. Let $m_0, m_1 > 0$ and $q > 0$ be positive real numbers. Let $\omega \in (-m_0, +m_0)$ and $p = 2^\star$. Assume

$$m_0^2 < \omega^2 + \frac{n-2}{4(n-1)} S_g \tag{1}$$

in M, where S_g is the scalar curvature of g. Then (S_ω) possesses a smooth positive mountain pass solution. Moreover, there also holds that for any $\theta \in (0, 1)$, there exists $C > 0$ such that $\|\mathcal{U}\|_{C^{2,\theta}} \le C$ for all $\mathcal{U} \in S_{2^\star}(\omega')$ and all $\omega' \in (-m_0, +m_0) \backslash (-|\omega|, +|\omega|)$.

Concerning existence when $n = 4$ we just need to have (1) at one point in M. The problem is local in that case. When $n = 3$ we may replace the scalar curvature term by the maximum potential term for which we do have positivity of the mass.

Consequence 1 :
Whatever m_0 is, there exists $\varepsilon_0 > 0$ such that we do get existence and a priori bounds in the range $m_0^2 - \varepsilon_0 < \omega^2 < m_0^2$ (phase compensation).

Consequence 2 :
In case $m_0^2 < \frac{n-2}{4(n-1)} S_g$, we do get existence and a priori bounds for all phases, and thus for the full range of phases.

Here again we prevent the existence of standing waves with arbitrarily large amplitudes (e.g., when $m_0 \gg 1$, fast oscillating standing waves cannot have arbitrarily large amplitudes).

When $m_0 \gg 1$, Theorem 1 answers our question for large ω's, and we are left with the question for the other values of ω, namely when $\omega^2 \leq m_0^2 - \frac{n-2}{4(n-1)} S_g$. Here the answer depends strongly on the dimension.

Theorem 2 (3-dim resonant states ; H.-Wei, 2012)

Let $(\mathbb{S}^3, g)$ be the unit 3-sphere, $m_0, m_1 > 0$, and $q > 0$. Let $p = 2^\star$. There exists a sequence $(\theta_k)_k$ of positive real numbers, satisfying that $\theta_1 = \frac{\sqrt{3}}{2}$, $\theta_k > \theta_1$ when $k \geq 2$, and $\theta_k \to +\infty$ as $k \to +\infty$, and there exists a sequence $(c_k(m_1))_k$, satisfying that $c_1(m_1) = 0$, $c_k(m_1) > 0$ for $k \geq 2$, and $c_k(m_1) \to +\infty$ as $k \to +\infty$, such that any $\omega_k \in (-m_0, m_0)$ given by $\theta_k^2 = m_0^2 - \omega_k^2$, which satisfy that $q^2\omega_k^2 \neq c_k(m_1)$, is an resonant phase for (S_ω) associated with a k-spikes configuration.

For any such ω_k, there exists $(u_\alpha e^{-i\omega_\alpha t})_\alpha$ and $(v_\alpha)_\alpha$ solutions of

$$\begin{cases} \Delta_g u_\alpha + m_0^2 u_\alpha = u_\alpha^{2^\star-1} + \omega_\alpha^2 \left(qv_\alpha - 1\right)^2 u_\alpha \\ \Delta_g v_\alpha + \left(m_1^2 + q^2 u_\alpha^2\right) v_\alpha = q u_\alpha^2 \end{cases} \qquad (S_\alpha)$$

for all α, such that $\omega_\alpha \to \omega_k$ and $\|u_\alpha\|_{L^\infty} + \|v_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$ (and k bubbles are involved in the construction).

The condition $q^2\omega_k^2 \neq c_k(m_1)$ is automatically satisfied when we require that $qm_0 \ll m_1$.

Theorem 3 (4-dimensional phase stability ; Druet-H.-Vétois, 2013)

Let (M,g) be a smooth compact Riemannian manifold of dimensions $n=4$. Let $m_0, m_1 > 0$ and $q > 0$ be positive real numbers. Let $\omega \in (-m_0, +m_0)$, and $p = 2^\star$. Assume

$$m_0^2 - \omega^2 \notin \left[\frac{1}{6}\min_M S_g, \frac{1}{6}\max_M S_g\right]$$

Then ω is a stable phase for (S_ω). Conversely, on the standard sphere $(\mathbb{S}^4, g)$, when $m_0^2 \geq \frac{1}{6}\max_M S_g$, the two $\pm\omega$'s given by the equation $m_0^2 - \omega^2 = \frac{1}{6} S_g$ are resonant phases for (S_ω).

N.B. : the first part of the result holds true even when S_g is not positive. In particular all phases are stable when $S_g \leq 0$ in M (like in the model cases of flat torii and compact hyperbolic spaces).

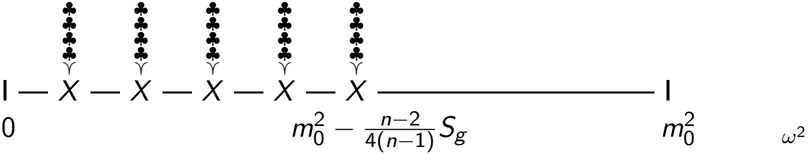
The first part of the theorem is false when $n = 3$ by the preceding theorem (establishing the existence of a whole family of resonant states for small ω's).

Summarizing in the $\mathbb{S}^3$ and $\mathbb{S}^4$ model cases :

[1] Case of $\mathbb{S}^3$: $\left(\frac{n-2}{4(n-1)} S_g = \frac{3}{4}\right)$

Resonant States[Thm2] A priori bounds[Thm1]

(No resonant states)

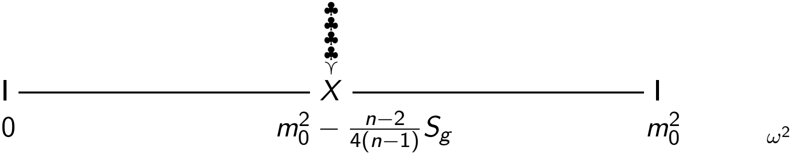


[2] Case of $\mathbb{S}^4$: $\left(\frac{n-2}{4(n-1)} S_g = 2\right)$

Phase stability[Thm3] A priori bounds[Thm1]

(No resonant states) (No resonant states)

Resonant states[Thm3]



V. Proof of the results :

+ Variational analysis based on the mountain pass lemma for the existence of solutions (in the spirit of Ambrosetti-Rabinowitz, Aubin, Benci-Fortunato, Brézis-Nirenberg, Schoen).

+ Constructive approach and the finite dimensional reduction argument for the resonant states part of the results (in the spirit of the stationnary Schrödinger constructions by Brendle, Del Pino, Malchiodi, Marques, Mazzeo, Micheletti, Pacard, Pistoia, Rey, Robert, Vétois, Wei).

+ Apriori analysis based on the Liouville obstructions (as in Gidas-Spruck) for the subcritical case, apriori analysis based on the bounded stability approach (in the spirit of the Yamabe compactness proofs by Druet, Li-Zhang, Li-Zhu, Marques, Khuri-Marques-Schoen, Schoen) for the apriori bounds part of the theorems, and apriori analysis based on the C^0-theory for blow-up (by Druet-H.-Robert) and the notion of the interaction range of bubbles (by Druet) for the stable phases part of the results.

VI. Proof of the results (existence)

Natural energy functional :

$$S(u,v) = \frac{1}{2}\int_M |\nabla u|^2 dv_g - \frac{\omega^2}{2}\int_M |\nabla v|^2 dv_g + \frac{m_0^2}{2}\int_M u^2 dv_g$$
$$-\frac{\omega^2 m_1^2}{2}\int_M v^2 dv_g - \frac{1}{p}\int_M u^p dv_g - \frac{\omega^2}{2}\int_M u^2(1-qv)^2 dv_g \ .$$

Define $\Phi : H^1 \to H^1$ by

$$\Delta_g \Phi(u) + \left(m_1^2 + q^2 u^2\right)\Phi(u) = qu^2 \ .$$

We can prove that Φ is differentiable when $n = 3, 4$. Define I_p by

$$I_p(u) = \frac{1}{2}\int_M |\nabla u|^2 dv_g + \frac{m_0^2}{2}\int_M u^2 dv_g - \frac{1}{p}\int_M (u^+)^p dv_g$$
$$-\frac{\omega^2}{2}\int_M \left(1 - q\Phi(u)\right) u^2 dv_g \ .$$

The critical points of I_p are the solutions of (S_ω).

The mountain pass lemma applies directly when $p < 2^\star$. When $p = 2^\star$, we follow the Aubin-Brézis-Nirenberg scheme and want to prove that

$$\inf_{P \in \mathcal{S}} \max_{u \in P} I_{2^\star}(u) < \frac{1}{n K_n^n}$$

where $\mathcal{S}$ is the set of continuous paths from 0 to u_0, $\|u_0\|_{H^1} \gg 1$, and K_n is the sharp constant in $\|u\|_{L^{2^\star}} \leq K_n \|\nabla u\|_{L^2}$, $u \in \dot{H}^1(\mathbb{R}^n)$. We use the Schoen's test functions when $n = 3$, and the Aubin's test functions when $n = 4$. Phase compensation comes with the estimate

$$\int_M \Phi(u_\varepsilon) u_\varepsilon^2 dv_g = o\left(\int_M u_\varepsilon^2 dv_g\right) .$$

E.g., when $n = 3$, $\|\Phi(u_\varepsilon)\|_{L^\infty} = o(1)$. When $n \geq 5$ this estimate does not hold true anymore. Namely,

$$\frac{\int_M \Phi(u_\varepsilon) u_\varepsilon^2 dv_g}{\int_M u_\varepsilon^2 dv_g} = \frac{1}{q} + o(1)$$

when $n \geq 5$ (the second equation in (S_ω) is supercritical in these dimensions).

VII. Proof of the results (compactness part)

We let $u_\alpha e^{i\omega_\alpha t}$ be arbitrary standing wave solutions of the equations (S_α) with electric fields $v_\alpha = \Phi(u_\alpha)$. We assume that $\|u_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$.

(i) For the stable a priori bound property we let x_α where u_α is maximum. The u_α's develop a bubble at x_α. We let r_α be the range up to which the bubble is leading. We prove that $r_\alpha \not\to 0$ (blow-up points are isolated), and then eliminate such blow-up points. Essentially we follow the blow-up analysis developed to prove compactness of the Yamabe problem. It's a one bubble analysis. The potential here is given by

$$h_\alpha = m_0^2 - \omega_\alpha^2 \left(q\Phi(u_\alpha) - 1\right)^2$$

and is only controlled in L^∞. Makes essentially no difference when $n = 3$, but makes the analysis more involved when $n = 4$. The blow-up analysis leads to a contradiction when $\Delta_g + (m_0^2 - \omega^2)$ has a positive mass and $n = 3$, or when $m_0^2 - \omega^2 < \frac{n-2}{4(n-1)} \min S_g$ and $n = 4$ (arguments go back to Druet, Li-Zhu, Marques, Schoen).

(ii) For the phase stability part we need to pay attention to the interaction of bubbles. Here $n = 4$. We assume that the u_α's have bounded energy. By the H^1-theory (Struwe), up to a subsequence,

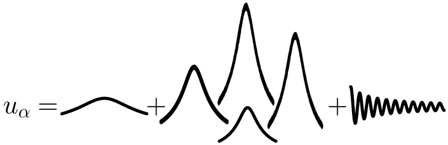
$$u_\alpha = u_\infty + \sum_{i=1}^{k} B_\alpha^i + R_\alpha \ ,$$

where $\left(u_\infty e^{i\omega t}, \Phi(u_\infty)\right)$ solve the limit system (S_ω), the $(B_\alpha^i)_\alpha$'s are bubbles, and $R_\alpha \to 0$ in H^1 as $\alpha \to +\infty$. The equation for a bubble is like

$$B_\alpha^i(x) = \left(\frac{\mu_{i,\alpha}}{\mu_{i,\alpha}^2 + \frac{d_g(x_{i,\alpha},x)^2}{n(n-2)}}\right)^{\frac{n-2}{2}}$$

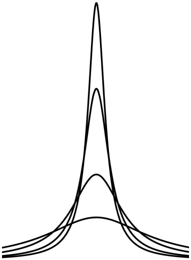
where $\mu_{i,\alpha} \to 0$ as $\alpha \to +\infty$ and, up to a subsequence, the $x_{i,\alpha}$'s converge in M (the limit x_i of the $x_{i,\alpha}$'s is said to be a geometric blow-up point for u_α). The set of geometric blow-up points may be reduced to one point, even when $k \gg 1$, due to the accumulation of bubbles.

On a drawing, the H^1-decomposition is represented by



$$= \quad u_\infty \quad + \quad \sum_{i=1}^{k} B_\alpha^i \quad + \quad R_\alpha$$

and the evolution of a bubble is given by

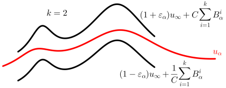


The C^0-theory (Druet-H.-Robert) gives that there exists $C > 1$ and a sequence $(\varepsilon)_\alpha$ of positive real numbers converging to zero such that, up to a subsequence,

$$(1-\varepsilon_\alpha)u_\infty(x) + \frac{1}{C}\sum_{i=1}^{k} B_\alpha^i(x)$$

$$\leq u_\alpha(x) \leq (1+\varepsilon_\alpha)u_\infty(x) + C\sum_{i=1}^{k} B_\alpha^i(x)$$

for all $x \in M$ and all α. In other words, for instance when $k = 2$,

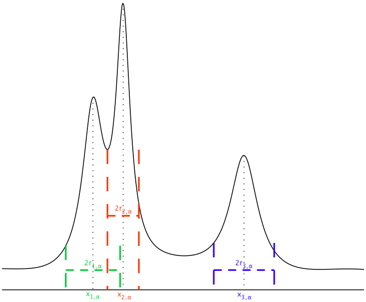


We even have that

$$u_\alpha = \big(1+o(1)\big)u_\infty + \sum_{i=1}^{k}\big(\Phi(x_i,\cdot)+o(1)\big)B_\alpha^i\ ,$$

where $\Phi : M \times M \to \mathbb{R}$ is continuous and equals one on the diagonal.

We define the range of interaction of a given bubble $(B^i_\alpha)_\alpha$, namely the range $r_{i,\alpha}$ up to which the bubble is leading and at which it starts interacting with another bubble. In doing so we forget about the much higher bubbles. On a drawing (here $u_\infty \not\equiv 0$, $\mu_{2,\alpha} = o(\mu_{1,\alpha})$, and $d_g(x_{1,\alpha}, x_{2,\alpha}) = o(\mu_{1,\alpha})$),



Then $r_{1,\alpha} = \sqrt{\mu_{1,\alpha}}$, $r_{3,\alpha} = \sqrt{\mu_{3,\alpha}}$, and $r_{2,\alpha} \sim \sqrt{\mu_{1,\alpha}\mu_{2,\alpha}}$. We can check, $B^1_\alpha(x_\alpha) = B^2_\alpha(x_\alpha)$ if and only if $d_g(x_{2,\alpha}, x_\alpha) \sim r_{2,\alpha}$.

Let $\mu_\alpha = \max_j \mu_{j,\alpha}$. By the C^0-theory, assuming that $r_{i,\alpha} = o\left(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\right)$, we get that

$$r_{i,\alpha}^{n-2}\mu_{i,\alpha}^{1-\frac{n}{2}} u_\alpha\left(\exp_{x_{i,\alpha}}(r_{i,\alpha}x)\right) \to \frac{(n(n-2))^{\frac{n-2}{2}}}{|x|^{n-2}} + \mathcal{H}_i(x)$$

in $C^2_{loc}(B_0(\delta)\backslash\{0\})$ as $\alpha \to +\infty$ for $0 < \delta \ll 1$, where $\mathcal{H}_i$ is a harmonic function in $B_0(\delta)$ satisfying that $\mathcal{H}_i(0) \neq 0$ and $\nabla\mathcal{H}_i(0) \equiv 0$. It turns out that we have an explicit expression for $\mathcal{H}_i$. In particular, when $\mu_{i,\alpha} \sim \max_j \mu_{j,\alpha}$, then

$$\mathcal{H}_i(x) = \sum_{j\in I_i} \frac{\lambda_{i,j}}{|x - x_{i,j}|^{n-2}} + \left(\lim_{\alpha\to+\infty} r_{i,\alpha}^{n-2}\mu_{i,\alpha}^{1-\frac{n}{2}}\right) u_\infty(x_i) , \qquad (\star)$$

where x_i is the limit of the $x_{i,\alpha}$'s,$\lambda_{i,j} > 0$ for all i, j, the $x_{i,j}$'s are given by

$$x_{i,j} = \lim_{\alpha\to+\infty} \frac{1}{r_{i,\alpha}} \exp^{-1}_{x_{i,\alpha}}(x_{j,\alpha}) ,$$

and I_i essentially consists of the j's such that $\mu_{j,\alpha} \sim \mu_{i,\alpha}$.

Apply the Pohozaev identity around the $x_{i,\alpha}$'s at a scale like $r_{i,\alpha}$. We get that

$$\begin{aligned}&\left(h_\infty(x_i) - \frac{1}{6}S_g(x_i) + o(1)\right)\mu_{i,\alpha}^2 \ln \frac{r_{i,\alpha}}{\mu_{i,\alpha}}\\ &= O\left(\mu_{i,\alpha}\mu_\alpha\right) + o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) \quad \text{in general },\\ &= \Lambda\left(\mathcal{H}_i(0) + o(1)\right)\mu_{i,\alpha}^2 r_{i,\alpha}^{-2} + o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right)\end{aligned}$$

when $r_{i,\alpha} = o\left(\sqrt{\mu_{i,\alpha}/\mu_\alpha}\right)$. Here $h_\alpha \to h_\infty$, $h_\infty \in L^\infty$. From the definition of the range of influence, $r_{i,\alpha} \leq \sqrt{\mu_{i,\alpha}}$ when $u_\infty \not\equiv 0$. In particular, $r_{i,\alpha} \to 0$ as $\alpha \to +\infty$. Now if we pick i such that $\mu_{i,\alpha} \sim \max_j \mu_{j,\alpha}$, then the Pohozaev expansion gives that

$$\mathcal{H}_i(0)\mu_{i,\alpha}^2 r_{i,\alpha}^{-2} = O\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) .$$

Since $\mu_{i,\alpha}^2 r_{i,\alpha}^{-2} \geq C\mu_{i,\alpha}$, we would get that $\mathcal{H}_i(0) = 0$, a contradiction with $\mathcal{H}_i(0) \neq 0$. This proves that $u_\infty \equiv 0$.

Still from the Pohozaev expansions,

$$\left(h_\infty(x_i) - \frac{1}{6}S_g(x_i)\right)\mu_{i,\alpha}^2 \ln \frac{r_{i,\alpha}}{\mu_{i,\alpha}} = O\left(\mu_{i,\alpha}\mu_{1,\alpha}\right) + o\left(\mu_{i,\alpha}^2 \ln \frac{1}{\mu_{i,\alpha}}\right) .$$

Since $u_\infty \equiv 0$, we get that $\Phi(u_\infty) \equiv 0$, and there holds that

$$h_\infty = m_0^2 - \omega^2 .$$

So when we assume that

$$m_0^2 - \omega^2 \notin \left[\frac{1}{6}\min_M S_g, \frac{1}{6}\max_M S_g\right] ,$$

then $r_{i,\alpha} \to 0$ for all i such that $\mu_{i,\alpha} \sim \max_j \mu_{j,\alpha}$. Let I_1 be the set of such i's, and let $i \in I_1$ be such that

$$d_g(x_{1,\alpha}, x_{i,\alpha}) \geq d_g(x_{1,\alpha}, x_{j,\alpha})$$

for all $j \in I_1$. Then the $x_{i,j}$'s all lie in a ball in the Euclidean space whose boundary contains 0, and they are not 0. In particular, for this i, there exists a vector $\nu_i \in \mathbb{R}^4$ such that ν_i has norm one and $\langle x_{i,j}, \nu_i\rangle > 0$ for all $j \in I_1$.

The expression for $\mathcal{H}_i$ was

$$\mathcal{H}_i(x) = \sum_{j \in I_i} \frac{\lambda_{i,j}}{|x - x_{i,j}|^{n-2}} + \left(\lim_{\alpha \to +\infty} r_{i,\alpha}^{n-2} \mu_{i,\alpha}^{1-\frac{n}{2}} \right) u_\infty(x_i) \ .$$

Then we get that

$$\nabla \mathcal{H}_i(0).\nu_i = \sum_j \frac{\lambda_{i,j} \langle x_{i,j}, \nu_i \rangle}{|x_{i,j}|^n}$$

and since $u_\infty \equiv 0$, we obtain that

$$\nabla \mathcal{H}_i(0) \equiv 0 \Rightarrow \mathcal{H}_i(0) = 0 \ ,$$

a contradiction. This proves Theorem 3.

VIII. A short physics break (bis)

-Few lines by Louis de Broglie (1950)

-Few lines by Eric Adelberger, Gia Dvali, Andrei Gruzinov (2007)

Louis de Broglie, 1950
Sur une forme nouvelle de la théorie du champ soustractif

A partir de 1934, l'auteur du présent article a développé une forme nouvelle de la théorie quantique du champ électromagnétique qu'il a appelé "la Mécanique ondulatoire du photon" et qui présentait à ses yeux l'avantage de faire plus clairement rentrer la théorie quantique des champs dans le cadre général de la Mécanique ondulatoire des particules à spin. Dans cette théorie, qui a été exposée dans plusieurs Ouvrages, il a été attribué au photon une masse propre extrêmement petite, mais non nulle, et nous avons été ainsi conduit dès 1934 à prendre comme équations de la particule de spin 1 des équations qui, mises sous forme vectorielle, sont des équations du type classique de Maxwell complétées par de petits termes contenant la masse propre. Des équations de même forme ont été ensuite proposées, en 1936, par M. Alexandre Proca, et on leur donne aujourd'hui dans la théorie du méson le nom d'équations de Proca. En somme ces équations sont les équations générales des particules de spin 1.

Eric Adelberger, Gia Dvali, Andrei Gruzinov, 2007
Photon-mass bound destroyed by vortices

The possibility of a nonzero photon mass remains one of the most important issues in physics, as it would shed light on fundamental questions such as charge conservation, charge quantization, the possibility of charged black holes and magnetic monopoles, etc. The most stringent upper bounds on the photon mass listed by the particule Data Group, $m < 3 \times 10^{-27}$ eV and $m < 2 \times 10^{-16}$ eV, are based on the assumption that a massive photon cause large-scale magnetic fields to be accompanied by an energy density

$$m_A^2 \tilde{A}_\mu \tilde{A}^\mu$$

associated with the Proca field $\tilde{A}_\mu$ that describes the massive photon. [...]

X. References for our results.

References

[1] O.Druet and E.Hebey, Existence and a priori bounds for electrostatic Klein-Gordon-Maxwell systems in fully inhomogeneous spaces, *Commun. Contemp. Math.*, 12, 2010, 831–869.

[2] O.Druet, E.Hebey, J.Vétois, Static Klein-Gordon-Maxwell-Proca systems in 4-dimensional closed manifolds .II, *J. reine angew. Math.*, to appear.

[3] E.Hebey and T. Truong, Static Klein-Gordon-Maxwell-Proca systems in 4-dimensional closed manifolds, *J. reine angew. Math.*, 667, 2012, 221–248.

[4] E.Hebey and J.Wei, Resonant states for the static Klein-Gordon-Maxwell-Proca system, *Math. Res. Lett.*, 19, 2012, 953–967.