Existence, stability and instability for Einstein-scalar field Lichnerowicz equations

by

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Contents:

Given Ψ scalar field, and $V(\Psi)$ a potential, Einstein-scalar field equations are written as:

$$G_{ij} = \nabla_i \Psi \nabla_j \Psi - \frac{1}{2} \left(\nabla^\alpha \Psi \nabla_\alpha \Psi \right) \gamma_{ij} - V(\Psi) \gamma_{ij} \ ,$$

where γ is the spacetime metric, and $G = Rc_\gamma - \frac{1}{2} S_\gamma \gamma$ is the Einstein curvature tensor. In the massive Klein-Gordon field theory,

$$V(\Psi) = \frac{1}{2} m^2 \Psi^2 \ .$$

The constraint equations, using the conformal method, are

$$\frac{4(n-1)}{n-2}\Delta_g u + h(g,\psi)u = f(\psi,\tau)u^{2^\star-1} + \frac{a(\sigma,W,\pi)}{u^{2^\star+1}} \ , \quad (1)$$
$$\mathrm{div}_g(\mathcal{D}W) = \frac{n-1}{n}u^{2^\star}\nabla\tau - \pi\nabla\psi \ , \quad (2)$$

where $\Delta_g = -\mathrm{div}_g\nabla$, $2^\star = 2n/(n-2)$,

$$h = S_g - |\nabla\psi|^2 \ , \quad a = |\sigma + \mathcal{D}W|^2 + \pi^2 \ , \ f = 4V(\psi) - \frac{n-1}{n}\tau^2$$

and S_g is the scalar curvature of g. Here, ψ, π and τ are functions connected to the physics setting (τ mean curvature of spacelike hypersurface), σ TT-tensor, W vector field, and $\mathcal{D}$ the conformal Killing operator given by

$$(\mathcal{D}W)_{ij} = (\nabla_i W)_j + (\nabla_j W)_i - \frac{2}{n}(\mathrm{div}_g W)g_{ij} \ .$$

The system $(1) - (2)$ is decoupled in the constant mean curvature setting, namely when $\tau = C^{te}$.

The free data are $(g, \sigma, \tau, \psi, \pi)$. The determined data are u and W. They satisfy

$$\frac{4(n-1)}{n-2}\Delta_g u + h(g,\psi)u = f(\psi,\tau)u^{2^\star-1} + \frac{a(\sigma,W,\pi)}{u^{2^\star+1}} \ , \quad (1)$$

$$\mathrm{div}_g(\mathcal{D}W) = \frac{n-1}{n}u^{2^\star}\nabla\tau - \pi\nabla\psi \ . \quad (2)$$

More details available in the survey paper by Robert Bartnik and Jim Isenberg: *The constraint equations*, arXiv:gr-qc/0405092v1, 2004.

(M,g) compact, $\partial M = \emptyset$, $n \geq 3$. Let h, a, and f be arbitrary smooth functions in M. Assume $a > 0$. Consider

$$\Delta_g u + hu = fu^{2^\star - 1} + \frac{a}{u^{2^\star + 1}} \ , \tag{EL}$$

where $\Delta_g = -\mathrm{div}_g \nabla$, and $2^\star = \frac{2n}{n-2}$.

Example: (Sub and supersolution method, Choquet-Bruhat, Isenberg, Pollack, 2006). Assume $\Delta_g + h$ is coercive and $f \leq 0$. Let $v > 0$ and $u_0 > 0$ be such that

$$\Delta_g u_0 + hu_0 = v \ .$$

For $t > 0$, let $u_t = tu_0$. We have:

(i) u_t is a subsolution of (EL) when $t \ll 1$, and

(ii) u_t is a supersolution of (EL) when $t \gg 1$.

Since $u_t \leq u_{t'}$ when $t \leq t'$, the sub and supersolution method provides a solution "$u \in [u_t, u_{t'}]$" for (EL).

Question: *What can we say when* $\Delta_g + h$ *is coercive and either* f *changes sign or* f *is everywhere positive, i.e. when* $\max_M f > 0$ *?*

Assume $\Delta_g + h$ is coercive. Define

$$\|u\|_h^2 = \int_M \left(|\nabla u|^2 + hu^2\right) dv_g \ ,$$

$u \in H^1$. Let $S(h)$ to be the smallest constant such that

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq S(h)^{2/2^\star} \int_M \left(|\nabla u|^2 + hu^2\right) dv_g$$

for all $u \in H^1$.

Theorem 1: (H.-Pacard-Pollack, Comm. Math. Phys., 2008) *Let* (M, g) *be a smooth compact Riemannian manifold,* $n \geq 3$*. Let* h*,* a*, and* f *be smooth functions in* M*. Assume that* $\Delta_g + h$ *is coercive, that* $a > 0$ *in* M*, and that* $\max_M f > 0$*. There exists* $C = C(n)$*,* $C > 0$ *depending only on* n*, such that if*

$$\|\varphi\|_h^{2^\star} \int_M \frac{a}{\varphi^{2^\star}} dv_g < \frac{C(n)}{(S(h) \max_M |f|)^{n-1}}$$

and $\int_M f \varphi^{2^\star} dv_g > 0$ *for some smooth positive function* $\varphi > 0$ *in* M*, then the Einstein-scalar field Lichnerowicz equation* (EL) *possesses a smooth positive solution.*

Example: if $\int_M f dv_g > 0$ then take $\varphi \equiv 1$ and the condition reads as

$$\int_M a dv_g < \frac{C(n, g, h)}{(\max_M |f|)^{n-1}} ,$$

where $C(n, g, h) > 0$ depends on n, g and h.

A perturbation of (EL) is a sequence $(EL_\alpha)_\alpha$ of equations, $\alpha \in \mathbb{N}$, which are written as

$$\Delta_g u + h_\alpha u = f_\alpha u^{2^\star - 1} + \frac{a_\alpha}{u^{2^\star + 1}} + k_\alpha \qquad (EL_\alpha)$$

for all α. Here we require that

$$h_\alpha \to h \ , \ a_\alpha \to a \ , \ k_\alpha \to 0$$

in C^0 as $\alpha \to +\infty$, and that $f_\alpha \to f$ in $C^{1,\eta}$ as $\alpha \to +\infty$, where $\eta > \frac{1}{2}$.

If (EL) satisfies the assumption of Theorem 1, any perturbation of (EL) also satisfies the assumptions of Theorem 1.

A sequence $(u_\alpha)_\alpha$ is a sequence of solutions of $(EL_\alpha)_\alpha$ if for any α, u_α solves (EL_α).

Definition: (Elliptic stability) *The Einstein-scalar field Lichnerowicz equation* (EL) *is said to be:*

(i) stable if for any perturbation $(EL_\alpha)_\alpha$ *of* (EL)*, and any* H^1*-bounded sequence* $(u_\alpha)_\alpha$ *of smooth positive solutions of* $(EL_\alpha)_\alpha$*, there exists a smooth positive solution* u *of* (EL) *such that, up to a subsequence,* $u_\alpha \to u$ *in* $C^{1,\theta}(M)$ *for all* $\theta \in (0,1)$*, and*

(ii) bounded and stable if for any perturbation $(EL_\alpha)_\alpha$ *of* (EL)*, and any sequence* $(u_\alpha)_\alpha$ *of smooth positive solutions of* $(EL_\alpha)_\alpha$*, the sequence* $(u_\alpha)_\alpha$ *is bounded in* H^1 *and there exists a smooth positive solution* u *of* (EL) *such that, up to a subsequence,* $u_\alpha \to u$ *in* $C^{1,\theta}(M)$ *for all* $\theta \in (0,1)$*.*

Remark 1: Assuming stronger convergences for the h_α's, f_α's, etc., then we get stronger convergences for the u_α's. E.g., if $h_\alpha \to h$, $f_\alpha \to f$, $a_\alpha \to a$ and $k_\alpha \to 0$ in $C^{p,\theta}$, $p \in \mathbb{N}$ and $\theta \in (0,1)$, then $u_\alpha \to u$ in $C^{p+2,\theta'}$, $\theta' < \theta$.

Remark 2: Stability means that if you slightly perturb h, a, and f, and even if you add to the equation a small "background noise" represented by k, then, in doing so, you do note create solutions which stand far from a solution of the original equation.

Remark 3: Say (EL) is compact if any H^1-bounded sequence $(u_\alpha)_\alpha$ of solutions of (EL) does possess a subsequence which converges in C^2. Say (EL) is bounded and compact if any sequence $(u_\alpha)_\alpha$ of solutions of (EL) does possess a subsequence which converges in C^2. Stability implies compactness. Bounded stability implies bounded compactness.

Let $\mathcal{D} = C^\infty(M)^4$ and $\|\cdot\|_{\mathcal{D}}$ be given by

$$\|D\|_{\mathcal{D}} = \sum_{i=1}^{3} \|f_i\|_{C^{0,1}} + \|f_4\|_{C^{1,1}}$$

for all $D = (f_1, f_2, f_3, f_4) \in \mathcal{D}$. For $D = (h, a, k, f)$ in $\mathcal{D}$ consider

$$\Delta_g u + hu = fu^{2^\star - 1} + \frac{a}{u^{2^\star + 1}} + k \ . \qquad (EL')$$

If $D = (h, a, 0, f)$, then $(EL') = (EL)$. Let $\Lambda > 0$, $D = (h, a, k, f)$ in $\mathcal{D}$, and define

$$\mathcal{S}_{D,\Lambda} = \left\{u \text{ solution of } (EL') \text{ s.t. } \|u\|_{H^1} \leq \Lambda\right\},$$
$$\text{and } \mathcal{S}_D = \left\{u \text{ solution of } (EL')\right\}.$$

When $D = (h, a, 0, f)$ we recover solutions of (EL).

For $X, Y \subset C^2$ define

$$d_{C^2}^{\hookrightarrow}(X;Y) = \sup_{u \in X} \inf_{v \in Y} \|v - u\|_{C^2} \ .$$

By convention, $d_{C^2}^{\hookrightarrow}(X;\emptyset) = +\infty$ if $X \neq \emptyset$, and $d_{C^2}^{\hookrightarrow}(\emptyset; Y) = 0$ for all Y, including $Y = \emptyset$.

Let $D = (h, a, 0, f)$ be given.

Stability $\Leftrightarrow$ (EL) is compact and
$\forall \varepsilon > 0,\ \forall \Lambda > 0,\ \exists \delta > 0$ s.t. $\forall D' = (h', a', k', f') \in \mathcal{D}$,
$\|D' - D\|_{\mathcal{D}} < \delta \ \Rightarrow \ d_{C^2}^{\hookrightarrow}(\mathcal{S}_{D',\Lambda}; \mathcal{S}_{D,\Lambda}) < \varepsilon$.

Bounded stability $\Leftrightarrow$ (EL) is bounded compact and
$\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. $\forall D' = (h', a', k', f') \in \mathcal{D}$,
$\|D' - D\|_{\mathcal{D}} < \delta \ \Rightarrow \ d_{C^2}^{\hookrightarrow}(\mathcal{S}_{D'}; \mathcal{S}_{D}) < \varepsilon$.

Theorem 2: (Druet-H., Math. Z., 2008) *Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and $h, a, f \in C^\infty(M)$ be smooth functions in M with $a > 0$. Assume $n = 3, 4, 5$. Then the Einstein-scalar field Lichnerowicz equation*

$$\Delta_g u + hu = fu^{2^\star - 1} + \frac{a}{u^{2^\star + 1}} \qquad (EL)$$

is stable. The equation is even bounded and stable assuming in addition that $f > 0$ in M. On the contrary, (EL) is not anymore stable a priori when $n \geq 6$.

I. Further directions and comments - 1

Lemma 1: (H.-Pacard-Pollack, Comm. Math. Phys., 2008)
Assume $a \geq 0$*,* $f > 0$*, and*

$$\frac{n^n}{(n-1)^{n-1}} \left(\int_M a^{\frac{n+2}{4n}} f^{\frac{3n-2}{4n}} dv_g \right)^{\frac{4n}{n+2}} > \left(\int_M \frac{(h^+)^{\frac{n+2}{4}} dv_g}{f^{\frac{n-2}{4}}} \right)^{\frac{4n}{n+2}} \quad .$$

Then the Einstein-scalar field Lichnerowicz equation (EL) *does not possess solutions.*

In particular, for any h, and any $f > 0$, there exist a positive constante $C = C(n, g, h, f)$ such that if

$$\int_M a^{\frac{n+2}{4n}} dv_g \geq C \ ,$$

then (EL) does not possess solutions.

Proof: Integrating (EL),

$$\int_M f u^{2^\star - 1} dv_g + \int_M \frac{a dv_g}{u^{2^\star + 1}} = \int_M h u dv_g \ .$$

By Hölder's inequalities,

$$\int_M h u dv_g \leq \left(\int_M \frac{(h^+)^{\frac{n+2}{4}} dv_g}{f^{\frac{n-2}{4}}} \right)^{\frac{4}{n+2}} \left(\int_M f u^{2^\star - 1} dv_g \right)^{\frac{n-2}{n+2}} \ , \text{ and}$$

$$\int_M a^{\frac{n+2}{4n}} f^{\frac{3n-2}{4n}} dv_g \leq \left(\int_M f u^{2^\star - 1} dv_g \right)^{\frac{3n-2}{4n}} \left(\int_M \frac{a dv_g}{u^{2^\star + 1}} \right)^{\frac{n+2}{4n}} \ .$$

$$X + \left(\int_M a^{\frac{n+2}{4n}} f^{\frac{3n-2}{4n}} dv_g \right)^{\frac{4n}{n+2}} X^{1-n} \leq \left(\int_M \frac{(h^+)^{\frac{n+2}{4}} dv_g}{f^{\frac{n-2}{4}}} \right)^{\frac{4}{n+2}} ,$$

where

$$X = \left(\int_M f u^{2^\star - 1} dv_g \right)^{\frac{4}{n+2}} .$$

This implies

$$\frac{n^n}{(n-1)^{n-1}} \left(\int_M a^{\frac{n+2}{4n}} f^{\frac{3n-2}{4n}} dv_g \right)^{\frac{4n}{n+2}} \leq \left(\int_M \frac{(h^+)^{\frac{n+2}{4}} dv_g}{f^{\frac{n-2}{4}}} \right)^{\frac{4n}{n+2}} . \diamondsuit$$

II. Further directions and comments - 2

Fix h, a and f. Assume $\Delta_g + h$ is coercive, and $a, f > 0$. Let $t > 0$ and consider

$$\Delta_g u + hu = fu^{2^\star - 1} + \frac{ta}{u^{2^\star + 1}} \ . \qquad (EL_t)$$

According to Theorem 1 and the Lemma:

(i) (Theorem 1) for $t \ll 1$, (EL_t) possesses a solution,

(ii) (Lemma 1) for $t \gg 1$, (EL_t) does not possess any solution.

Assuming $n = 3, 4, 5$,

(iii) (Theorem 2) $(EL_t)_t$ is bounded and stable for $t \in [t_0, t_1]$,

where $0 < t_0 < t_1$.

Let $\Lambda > 0$. Define

$$\Omega_\Lambda = \left\{ u \in C^{2,\theta} \text{ s.t. } \|u\|_{C^{2,\theta}} < \Lambda \text{ and } \min_M u > \Lambda^{-1} \right\} .$$

Fix $t_0 \ll 1$ such that (EL_{t_0}) possesses a solution. Fix $t_1 \gg 1$ such that (EL_{t_1}) does not possess any solution. Assume $n = 3, 4, 5$. Define $F_t : \overline{\Omega_\Lambda} \to C^{2,\theta}$ by

$$F_t u = u - L^{-1} \left(f u^{2^\star - 1} + \frac{ta}{u^{2^\star + 1}} \right) ,$$

where $L = \Delta_g + h$, and $t \in [t_0, t_1]$. By (iii), there exists $\Lambda_0 > 0$ such that $F_t^{-1}(0) \subset \Omega_{\Lambda_0}$ for all $t \in [t_0, t_1]$. Then, by (ii),

$$\deg(F_{t_0}, \Omega_\Lambda, 0) = 0$$

for all $\Lambda \gg 1$. In particular, assuming that the solutions of the equations are nondegenerate, the solution in Theorem 1 needs to come with another solution.

III. Proof of Theorem 1

We aim in proving:

Let (M,g) be a smooth compact Riemannian manifold, $n \geq 3$. Let h, a, and f be smooth functions in M. Assume that $\Delta_g + h$ is coercive, that $a > 0$ in M, and that $\max_M f > 0$. There exists $C = C(n)$, $C > 0$ depending only on n, such that if

$$\|\varphi\|_h^{2^\star} \int_M \frac{a}{\varphi^{2^\star}} dv_g < \frac{C(n)}{\left(S(h) \max_M |f|\right)^{n-1}}$$

and $\int_M f\varphi^{2^\star} dv_g > 0$ for some smooth positive function $\varphi > 0$ in M, then the Einstein-scalar field Lichnerowicz equation

$$\Delta_g u + hu = fu^{2^\star - 1} + \frac{a}{u^{2^\star + 1}} \qquad (EL)$$

possesses a smooth positive solution.

Method: approximated equations, mountain pass analysis.

Fix $\varepsilon > 0$. Define

$$I^{(1)}(u) = \frac{1}{2}\int_M \left(|\nabla u|^2 + hu^2\right) dv_g - \frac{1}{2^\star}\int_M f(u^+)^{2^\star} dv_g ,$$

and

$$I_\varepsilon^{(2)}(u) = \frac{1}{2^\star}\int_M \frac{a dv_g}{(\varepsilon + (u^+)^2)^{2^\star/2}} ,$$

where $u \in H^1$. Let

$$I_\varepsilon = I^{(1)} + I_\varepsilon^{(2)} .$$

Let $\varphi > 0$ be as in Theorem 1. Assume $\|\varphi\|_h = 1$. The conditions in the theorem read as

$$\int_M \frac{a}{\varphi^{2^\star}} dv_g < \frac{C(n)}{(S(h)\max_M |f|)^{n-1}} \qquad (1)$$

and $\int_M f\varphi^{2^\star} dv_g > 0$.

Let $\Phi, \Psi : \mathbb{R}^+ \to \mathbb{R}$ be the functions given by

$$\Phi(t) = \frac{1}{2}t^2 - \frac{\max_M |f|}{2^\star} S(h) t^{2^\star} \ , \ \text{and}$$
$$\Psi(t) = \frac{1}{2}t^2 + \frac{\max_M |f|}{2^\star} S(h) t^{2^\star} \ .$$

These functions satisfy

$$\Phi\left(\|u\|_h\right) \le I^{(1)}(u) \le \Psi\left(\|u\|_h\right) \qquad (2)$$

for all $u \in H^1$. Let $t_1 > 0$ be such that Φ is increasing up to t_1 and decreasing after:

$$t_1 = \left(S(h) \max_M |f|\right)^{-(n-2)/4} \ .$$

Let $t_0 > 0$ be given by

$$t_0 = \sqrt{\frac{1}{2(n-1)}}\ t_1 \ .$$

Then

$$\Psi(t_0) \leq \frac{1}{2}\Phi(t_1) \tag{3}$$

and for $C \ll 1$ the condition in the theorem translates into

$$\frac{1}{2^\star}\int_M \frac{a}{(t_0\varphi)^{2^\star}} dv_g < \frac{1}{2}\Phi(t_1)\ . \tag{4}$$

Let $\rho = \Phi(t_1)$. Then, by (3) and (4),

$$I_\varepsilon(t_0\varphi) < \rho$$

and by

$$\Phi\left(\|u\|_h\right) \leq I^{(1)}(u) \leq \Psi\left(\|u\|_h\right)\ , \tag{2}$$

we can write that

$$I_\varepsilon(u) \geq \rho$$

for all u s.t. $\|u\|_h = t_1$.

We got that there exists $\rho > 0$ such that

$$I_\varepsilon(t_0\varphi) < \rho$$

and

$$I_\varepsilon(u) \geq \rho$$

for all u s.t. $\|u\|_h = t_1$. Also $t_1 > t_0$. Since $\int_M f\varphi^{2^\star}dv_g > 0$,

$$I_\varepsilon(t\varphi) \to -\infty$$

as $t \to +\infty$.

$\Rightarrow$ We can apply the mountain pass lemma.

Let $t_2 \gg 1$. Define

$$c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} I_\varepsilon(u)$$

where Γ is the set of continuous paths joining $t_0\varphi$ to $t_2\varphi$. The MPL provides a Palais-Smale sequence $(u_k^\varepsilon)_k$ such that

$$I_\varepsilon(u_k^\varepsilon) \to c_\varepsilon \quad \text{and} \quad I'_\varepsilon(u_k^\varepsilon) \to 0$$

as $k \to +\infty$. The sequence $(u_k^\varepsilon)_k$ is bounded in H^1. Up to a subsequence, $u_k^\varepsilon \rightharpoonup u_\varepsilon$ in H^1. Then u_ε satisfies

$$\Delta_g u_\varepsilon + h u_\varepsilon = f u_\varepsilon^{2^\star - 1} + \frac{a u_\varepsilon}{(\varepsilon + u_\varepsilon^2)^{\frac{2^\star}{2} + 1}}$$

In particular, u_ε is positive and smooth.

We can prove that the c_ε's are bounded independently of ε. In particular the family $(u_\varepsilon)_\varepsilon$ is bounded in H^1. Now we can pass to the limit as $\varepsilon \to 0$ because u_ε will never approach zero. Take $x_\varepsilon \in M$ such that $u_\varepsilon(x_\varepsilon) = \min_M u_\varepsilon$. Then $\Delta_g u_\varepsilon(x_\varepsilon) \le 0$ and

$$|h(x_\varepsilon)| + |f(x_\varepsilon)|\, u_\varepsilon(x_\varepsilon)^{2^\star-2} \ge \frac{a(x_\varepsilon)}{(\varepsilon + u_\varepsilon(x_\varepsilon)^2)^{\frac{2^\star}{2}+1}} .$$

This implies that there exists $\delta_0 > 0$ such that

$$\min_M u_\varepsilon \ge \delta_0$$

for all ε. If $u_\varepsilon \rightharpoonup u$ in H^1, then $u \ge \delta_0$ and u solves (EL). $\diamondsuit$

IV. Proof of the stability part in Theorem 2

We aim in proving:

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and $h, a, f \in C^\infty(M)$ be smooth functions in M with $a > 0$. Assume $n = 3, 4, 5$. Then the Einstein-scalar field Lichnerowicz equation

$$\Delta_g u + hu = fu^{2^\star - 1} + \frac{a}{u^{2^\star + 1}} \qquad (EL)$$

is stable, and even bounded and stable if $f > 0$ in M.

Method: blow-up analysis, sharp pointwise estimates.

Let $(EL_\alpha)_\alpha$ be a perturbation of (EL). Let also $(u_\alpha)_\alpha$ be a sequence of solutions of (EL_α). Consider

(H1A) $f > 0$ in M,

(H1B) $(u_\alpha)_\alpha$ is bounded in H^1,

(H2) $\exists \varepsilon_0 > 0$ s.t. $u_\alpha \geq \varepsilon_0$ in M for all α.

We claim that:

Stability Theorem: (Druet-H., Math. Z., 2008) *Let $n \leq 5$. Let $(EL_\alpha)_\alpha$ be a perturbation of (EL) and $(u_\alpha)_\alpha$ a sequence of solutions of $(EL_\alpha)_\alpha$. Assume (H1A) or (H1B), and we also assume (H2). Then the sequence $(u_\alpha)_\alpha$ is uniformly bounded in $C^{1,\theta}$, $\theta \in (0,1)$.*

By (H2),

$$|\Delta_g u_\alpha| \leq C u_\alpha^{2^\star - 1} ,$$

where $C > 0$ does not depend on α.

Proof of stability theorem: By contradiction. We assume that $\|u_\alpha\|_\infty \to +\infty$ as $\alpha \to +\infty$. We also assume (H1A) or (H1B), and (H2). Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be such that

(i) x_α is a critical point of u_α for all α,

(ii) $\rho_\alpha^{\frac{n-2}{2}} \sup_{B_{x_\alpha}(\rho_\alpha)} u_\alpha \to +\infty$ as $\alpha \to +\infty$, and

(iii) $d_g(x_\alpha, x)^{\frac{n-2}{2}} u_\alpha(x) \le C$ for all $x \in B_{x_\alpha}(\rho_\alpha)$ and all α.

Then :

Main Estimate: *Assume (i) – (iii). Then we have that* $\rho_\alpha \to 0$, $\rho_\alpha^{\frac{n-2}{2}} u_\alpha(x_\alpha) \to +\infty$, *and*

$$u_\alpha(x_\alpha)\rho_\alpha^{n-2} u_\alpha\left(\exp_{x_\alpha}(\rho_\alpha x)\right) \to \frac{\lambda}{|x|^{n-2}} + H(x)$$

in $C^2_{loc}(B_0(1)\backslash\{0\})$ *as* $\alpha \to +\infty$, *where* $\lambda > 0$ *and* H *is a harmonic function in* $B_0(1)$ *which satisfies that* $H(0) = 0$.

There exist $C > 0$, a sequence $(N_\alpha)_\alpha$ of integers, and for any α, critical points $x_{1,\alpha}, \ldots, x_{N_\alpha,\alpha}$ of u_α such that

$$\left(\min_{i=1,\ldots,N_\alpha} d_g\left(x_{i,\alpha}, x\right) \right)^{\frac{n-2}{2}} u_\alpha(x) \leq C \tag{1}$$

for all $x \in M$ and all α. We have $N_\alpha \geq 2$. Define

$$d_\alpha = \min_{1 \leq i < j \leq N_\alpha} d_g(x_{i,\alpha}, x_{j,\alpha})$$

and let the $x_{i,\alpha}$'s be such that $d_\alpha = d_g(x_{1,\alpha}, x_{2,\alpha})$. We have $d_\alpha \to 0$ as $\alpha \to +\infty$. Moreover,

$$d_\alpha^{\frac{n-2}{2}} u_\alpha(x_{1,\alpha}) \to +\infty \tag{2}$$

as $\alpha \to +\infty$.

Define $\tilde{u}_\alpha$ by

$$\tilde{u}_\alpha(x) = d_\alpha^{\frac{n-2}{2}} u_\alpha\left(\exp_{x_{1,\alpha}}(d_\alpha x)\right) ,$$

where $x \in \mathbb{R}^n$. Let $\tilde{v}_\alpha = \tilde{u}_\alpha(0)\tilde{u}_\alpha$. Then

$$|\Delta_{\tilde{g}_\alpha}\tilde{v}_\alpha| \leq \frac{C}{\tilde{u}_\alpha(0)^{2^\star-2}}\tilde{v}_\alpha^{2^\star-1} , \tag{3}$$

where $\tilde{g}_\alpha \to \delta$ as $\alpha \to +\infty$. Because of

$$d_\alpha^{\frac{n-2}{2}} u_\alpha(x_{1,\alpha}) \to +\infty , \tag{2}$$

$\tilde{u}_\alpha(0) \to +\infty$ as $\alpha \to +\infty$. Independently, by elliptic theory, for any $R > 0$,

$$\tilde{v}_\alpha \to G \text{ in } C^1_{loc}\left(B_0(R)\backslash\{\tilde{x}_i\}_{i=1,\dots,p}\right)$$

as $\alpha \to +\infty$, where, because of (3), G is nonnegative and harmonic in $B_0(R)\backslash\{\tilde{x}_i\}_{i=1,\dots,p}$.

Then,

$$G(x) = \sum_{i=1}^{p} \frac{\lambda_i}{|x - \tilde{x}_i|^{n-2}} + H(x) \ ,$$

where $\lambda_i > 0$ and H is harmonic without singularities. In particular, in a neighbourhood of 0,

$$G(x) = \frac{\lambda_1}{|x|^{n-2}} + \tilde{H}(x) \ .$$

By

$$\left(\min_{i=1,\ldots,N_\alpha} d_g\left(x_{i,\alpha}, x\right) \right)^{\frac{n-2}{2}} u_\alpha(x) \le C \tag{1}$$

$$d_\alpha^{\frac{n-2}{2}} u_\alpha(x_{1,\alpha}) \to +\infty \tag{2}$$

we can apply the main estimate with $x_\alpha = x_{1,\alpha}$ and $\rho_\alpha = \frac{d_\alpha}{10}$. In particular, $\tilde{H}(0) = 0$.

However,

$$\begin{aligned} G(x) &= \frac{\lambda_1}{|x|^{n-2}} + \frac{\lambda_2}{|x-\tilde{x}_2|^{n-2}} + \hat{H}(x) \\ &\geq 0 \end{aligned}$$

and

$$\tilde{H}(x) = \frac{\lambda_2}{|x-\tilde{x}_2|^{n-2}} + \hat{H}(x) \ .$$

By the maximum principle,

$$\hat{H}(0) \geq \min_{\partial B_0(R)} \hat{H}$$

and we get that

$$\tilde{H}(0) \geq \frac{\lambda_2}{|\tilde{x}_2|^{n-2}} - \frac{\lambda_1}{R^{n-2}} - \frac{\lambda_2}{(R-|\tilde{x}_2|)^{n-2}} \ .$$

By construction, $|\tilde{x}_2| = 1$. Choosing $R \gg 1$ sufficiently large, $\tilde{H}(0) > 0$. A contradiction. $\diamondsuit$

It remains to prove the stability part in theorem 2. We introduced

(H1A) $f > 0$ in M,

(H1B) $(u_\alpha)_\alpha$ is bounded in H^1,

(H2) $\exists \varepsilon_0 > 0$ s.t. $u_\alpha \geq \varepsilon_0$ in M for all α,

and we proved that

$$(H1A) \text{ or } (H1B), \text{ and } (H2) \Rightarrow C^{1,\theta} - \text{convergences}$$

for the u_α's solutions of perturbations of (EL). Let $(EL_\alpha)_\alpha$ be any perturbation of (EL), and $(u_\alpha)_\alpha$ be any sequence of solution of $(EL_\alpha)_\alpha$. It suffices to prove (H2). Let x_α be such that $u_\alpha(x_\alpha) = \min u_\alpha$. Then $\Delta_g u_\alpha(x_\alpha) \leq 0$ and we get that

$$h_\alpha(x_\alpha) \geq \frac{1}{u_\alpha(x_\alpha)} \left(\frac{a_\alpha(x_\alpha)}{u_\alpha(x_\alpha)^{2^\star+1}} + k_\alpha(x_\alpha) \right) + f_\alpha(x_\alpha) u_\alpha(x_\alpha)^{2^\star-2} .$$

In particular, $u_\alpha \geq \varepsilon_0 > 0$ and (H2) is satisfied. We can apply the stability theorem. This proves the stability part of Theorem 2.

V. Proof of the instability part in Theorem 2

We aim in proving:

When $n \geq 6$ the Einstein-scalar field Lichnerowicz equation

$$\Delta_g u + hu = fu^{2^\star - 1} + \frac{a}{u^{2^\star + 1}} \qquad (EL)$$

is not a priori stable.

Method: explicit constructions of examples.

A first construction.

Lemma 2: (Druet-H., Math. Z., 2008) *Let* (S^n, g_0) *be the unit sphere,* $n \geq 7$*. Let* $x_0 \in S^n$*. Let* a *and* u_0 *be smooth positive functions such that*

$$\Delta_{g_0} u_0 + \frac{n(n-2)}{4} u_0 = \frac{n(n-2)}{4} u_0^{2^\star - 1} + \frac{a}{u_0^{2^\star + 1}} \ .$$

There exist sequences $(h_\alpha)_\alpha$ *and* $(\Phi_\alpha)_\alpha$ *such that* $h_\alpha \to \frac{n(n-2)}{4}$ *in* $C^0(S^n)$, $\max_M \Phi_\alpha \to +\infty$ *and* $\Phi_\alpha \to 0$ *in* $C^2_{loc}(S^n \backslash \{x_0\})$ *as* $\alpha \to +\infty$*. In addition*

$$\Delta_{g_0} u_\alpha + h_\alpha u_\alpha = \frac{n(n-2)}{4} u_\alpha^{2^\star - 1} + \frac{a}{u_\alpha^{2^\star + 1}}$$

for all α*, where* $u_\alpha = u_0 + \Phi_\alpha$*.*

Proof of Lemma 2: Let φ_α be given by

$$\varphi_\alpha(x) = \left(\frac{\sqrt{\beta_\alpha^2 - 1}}{\beta_\alpha - \cos d_{g_0}(x_0, x)} \right)^{\frac{n-2}{2}} ,$$

where $\beta_\alpha > 1$ for all α and $\beta_\alpha \to 1$ as $\alpha \to +\infty$. The φ_α's satisfy

$$\Delta_{g_0}\varphi_\alpha + \frac{n(n-2)}{4}\varphi_\alpha = \frac{n(n-2)}{4}\varphi_\alpha^{2^\star-1} .$$

Let

$$u_\alpha = u_0 + \varphi_\alpha + \psi_\alpha ,$$

where ψ_α is such that

$$\begin{aligned} &\Delta_{g_0} u_0 + \Delta_{g_0}\varphi_\alpha + \Delta_{g_0}\psi_\alpha \\ &= \frac{n(n-2)}{4}(u_0 + \varphi_\alpha)^{2^\star-1} - \left(\frac{n(n-2)}{4} + \varepsilon_\alpha \right)(u_0 + \varphi_\alpha) \\ &\quad + \frac{a}{(u_0 + \varphi_\alpha)^{2^\star+1}} . \end{aligned}$$

We have $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$.

For any sequence $(x_\alpha)_\alpha$ of points in S^n,

$$|\psi_\alpha(x_\alpha)| = o\left(\left(\frac{(\beta_\alpha - 1)^{\frac{(n-2)}{2(n-4)}}}{(\beta_\alpha - 1) + d_{g_0}(x_0, x_\alpha)^2}\right)^{\frac{n-4}{2}}\right) + o(1)\ . \qquad (1)$$

Thanks to (1),

$$\frac{\psi_\alpha}{u_\alpha} \to 0 \ \text{ and } \ u_\alpha^{2^\star - 3}\psi_\alpha \to 0 \qquad (2)$$

in $C^0(S^n)$ as $\alpha \to +\infty$. For instance, either $\psi_\alpha(x_\alpha) \to 0$ and $\psi_\alpha(x_\alpha)/u_\alpha(x_\alpha) \to 0$, or $\psi_\alpha(x_\alpha) \not\to 0$. In that case, because of (1), $d_{g_0}(x_0, x_\alpha) \to 0$. Then, $\psi_\alpha(x_\alpha)/u_\alpha(x_\alpha) \to 0$ since

$$\frac{\psi_\alpha(x_\alpha)}{\varphi_\alpha(x_\alpha)} \le C^{te}\Big((\beta_\alpha - 1) + d_{g_0}(x_0, x_\alpha)^2\Big)\ .$$

Let h_α be such that

$$\Delta_{g_0} u_\alpha + h_\alpha u_\alpha = \frac{n(n-2)}{4} u_\alpha^{2^\star - 1} + \frac{a}{u_\alpha^{2^\star + 1}}$$

for all α. Write $\Delta_{g_0} u_\alpha = \Delta_{g_0} u_0 + \Delta_{g_0} \varphi_\alpha + \Delta_{g_0} \psi_\alpha$. By the equation satisfied by ψ_α,

$$\left(h_\alpha - \frac{n(n-2)}{4} \right) u_\alpha = O\left(u_\alpha^{2^\star - 2} \psi_\alpha \right) + O\left(\psi_\alpha \right) + \varepsilon_\alpha u_\alpha \ .$$

Divide by u_α, and conclude thanks to

$$\frac{\psi_\alpha}{u_\alpha} \to 0 \ \text{ and } \ u_\alpha^{2^\star - 3} \psi_\alpha \to 0 \tag{2}$$

that $h_\alpha \to \frac{n(n-2)}{4}$ in C^0 as $\alpha \to +\infty$. This proves Lemma 2. $\diamondsuit$

Say that (M, g) has a conformally flat pole at x_0 if g is conformally flat around x_0. Thanks to Lemma 2 we get:

Lemma 3: (Druet-H., Math. Z., 2008) *Let (M, g) be a smooth compact Riemannian manifold with a conformally flat pole, $n \geq 7$. There exists $\delta > 0$ such that the Einstein-scalar Lichnerowicz equation*

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = u^{2^\star - 1} + \frac{a}{u^{2^\star + 1}}$$

is not stable on (M, g) and possesses smooth positive solutions for all smooth functions $a > 0$ such that $\|a\|_1 < \delta$.

VI. The case $a \geq 0$. Unpublished result.

When $a > 0$ in M: let $u > 0$ be a solution of (EL). Let x_0 be such that $u(x_0) = \min_M u$. Then $\Delta_g u(x_0) \leq 0$ and

$$|h(x_0)|u(x_0) + |f(x_0)|u(x_0)^{2^\star - 1} \geq \frac{a(x_0)}{u(x_0)^{2^\star + 1}}$$

$\Rightarrow$ there exists $\varepsilon_0 = \varepsilon_0(h, f, a)$, $\varepsilon_0 > 0$, such that $u \geq \varepsilon_0$ in M.

Question: *Assume $\Delta_g + h$ is coercive, $a \geq 0$ and $\max_M f > 0$. What can we say when $Zero(a) \neq \emptyset$?*

In physics

$$a = |\sigma + DW|^2 + \pi^2 ,$$

where σ and π are free data, and W is the determined data given by the second equation in the system.

Recall (EL) is compact if any H^1-bounded sequence $(u_\alpha)_\alpha$ of solutions of (EL) does possess a subsequence which converges in C^2. Recall (EL) is bounded and compact if any sequence $(u_\alpha)_\alpha$ of solutions of (EL) does possess a subsequence which converges in C^2.

Theorem 3: (Druet, Esposito, H., Pacard, Pollack, Collected works - Unpublished, 2009) *Assume $\Delta_g + h$ is coercive, $a \geq 0$, $a \not\equiv 0$, and $\max_M f > 0$. Theorem 1 remains true without any other assumptions than those of Theorem 1. Assuming that $n = 3, 4, 5$, the equation is compact and even bounded and compact when $f > 0$.*

Existence follows from a combination of Theorem 1 and the sub and supersolution method. Compactness follows from the stability theorem in the proof of Theorem 2 together with an argument by Pierpaolo Esposito.

Proof of the existence part in Theorem 3: Assume the "assumptions" of Theorem 1 are satisfied: there exists $\varphi > 0$ such that

$$\|\varphi\|_h^{2^\star} \int_M \frac{a}{\varphi^{2^\star}} dv_g < \frac{C(n)}{\left(S(h) \max_M |f|\right)^{n-1}} \tag{1}$$

and $\int_M f \varphi^{2^\star} dv_g > 0$. Changing a into $a + \varepsilon_0$ for $0 < \varepsilon_0 \ll 1$, (1) is still satisfied, and since $a + \varepsilon_0 > 0$ we can apply Theorem 1. In particular,

(i) "$a \to a + \varepsilon_0$", $0 < \varepsilon_0 \ll 1$, and Theorem 1
$\Rightarrow \exists u_1$ a supersolution of (EL).

Now let $\delta > 0$ and let u_0 solve

$$\Delta_g u_0 + h u_0 = a - \delta f^-$$

For $\delta > 0$ sufficiently small, u_0 is close to the solution with $\delta = 0$, and since this solution is positive by the maximum principle, we get that $u_0 > 0$ for $0 < \delta \ll 1$. Fix such a $\delta > 0$.

Given $\varepsilon > 0$, let $u_\varepsilon = \varepsilon u_0$. Then

$$\Delta_g u_\varepsilon + h u_\varepsilon = \varepsilon a - \delta\varepsilon f^- \leq f u_\varepsilon^{2^\star - 1} + \frac{a}{u_\varepsilon^{2^\star + 1}}$$

provided $0 < \varepsilon \ll 1$. In particular,

(ii) $u_\varepsilon = \varepsilon u_0$, $0 < \varepsilon \ll 1$, is a subsolution of (EL).

Noting that $u_\varepsilon \leq u_1$ for $\varepsilon > 0$ sufficiently small, we can apply the sub and supersolution method and get a solution u to (EL) such that $u_\varepsilon \leq u \leq u_1$. ♢

The compactness part in Theorem 3 follows from the stability theorem in the proof of Theorem 2 together with the following result by Esposito which establishes the (H2) property of Druet and Hebey under general conditions.

We do not need in what follows the $C^{1,\eta}$-convergence of the f_α's. A C^0-convergence (and even less) is enough.

Lemma 4: (Esposito, Unpublished, 2009) *Let $n \leq 5$. Let $(EL_\alpha)_\alpha$ be a perturbation of (EL) and $(u_\alpha)_\alpha$ a sequence of solutions of $(EL_\alpha)_\alpha$. Assume $a_\alpha \geq 0$ in M for all α, and $a \not\equiv 0$. The (H2) property holds true: $\exists \varepsilon_0 > 0$ such that $u_\alpha \geq \varepsilon_0$ in M for all α.*

Proof of the lemma: Let $K > 0$ be such that $K + h_\alpha \geq 1$ in M for all α. Define $\tilde{h}_\alpha = K + h_\alpha$ and $\tilde{h} = K + h$. Let $\delta > 0$ and v_α^δ, v^δ, and r_α be given by

$$\Delta_g v_\alpha^\delta + \tilde{h}_\alpha v_\alpha^\delta = a_\alpha - \delta f_\alpha^- \ ,$$
$$\Delta_g v^\delta + \tilde{h} v^\delta = a - \delta f^- \ ,$$
$$\Delta_g r_\alpha + \tilde{h}_\alpha r_\alpha = k_\alpha \ .$$

There holds that $v_\alpha^\delta \to v^\delta$ in $C^0(M)$ as $\alpha \to +\infty$ and that $v^\delta \to v^0$ in $C^0(M)$ as $\delta \to 0$. By the maximum principle, $v^0 > 0$ in M. It follows that there exists $\delta > 0$ sufficiently small, and $\varepsilon_0 > 0$, such that $v_\alpha^\delta \geq \varepsilon_0$ in M for all $\alpha \gg 1$. Fix such a $\delta > 0$. Let $t > 0$

and define

$$w_\alpha = t v_\alpha^\delta + r_\alpha \ .$$

We have that $r_\alpha \to 0$ in $C^0(M)$ as $\alpha \to +\infty$. There exists $t_0 > 0$ such that

$$\begin{array}{rcl} \Delta_g w_\alpha + \tilde{h}_\alpha w_\alpha & = & t a_\alpha - t\delta f_\alpha^- + k_\alpha \\ & \leq & -f_\alpha^- w_\alpha^{2^\star - 1} + \dfrac{a_\alpha}{w_\alpha^{2^\star + 1}} + k_\alpha \end{array}$$

for all $0 < t < t_0$ and all $\alpha \gg 1$. As a consequence, since $a_\alpha \geq 0$ in M,

$$\begin{array}{l} \Delta_g (u_\alpha - w_\alpha) + \tilde{h}_\alpha (u_\alpha - w_\alpha) \\ \geq f_\alpha u_\alpha^{2^\star - 1} + f_\alpha^- w_\alpha^{2^\star - 1} + \dfrac{a_\alpha}{u_\alpha^{2^\star + 1}} - \dfrac{a_\alpha}{w_\alpha^{2^\star + 1}} \geq 0 \end{array}$$

for all $\alpha \gg 1$, at any point such that $u_\alpha - w_\alpha \leq 0$. The maximum principle then gives that $w_\alpha \leq u_\alpha$ in M for all $\alpha \gg 1$. Since $w_\alpha \geq \varepsilon_0$ in M for $\alpha \gg 1$, this ends the proof of the lemma. $\diamondsuit$