SUPER CRITICAL ENERGY SCALE INVARIANT EQUATIONS IN CRITICAL SPACES

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Abstract. We discuss an elementary direct proof of the nonexistence of nontrivial solutions of supercritical solutions in invariant scaling spaces.

Let $\dot{H}^2$ be the Sobolev space in $\mathbb{R}^n$ of functions such that $\Delta u \in L^2$ with the corresponding norm $\|u\|_{H^2}^2 = \int_{\mathbb{R}^n} (\Delta u)^2 dx$. We consider the equation

$$\begin{cases} \Delta u = |u|^{\frac{4}{n-4}} u \ \text{ in } \mathbb{R}^n \ , \\ u \in \dot{H}^2(\mathbb{R}^n) \ . \end{cases} \tag{0.1}$$

As is easily checked, the equation is supercritial with respect to the $\dot{H}^1$-control given by the Laplacian. The feature with (0.1) is that both $\|\cdot\|_{H^2}$ and (0.1) are invariant under the action of the scaling $u_\lambda(x) = \lambda^\alpha u(\lambda x)$, $\lambda > 0$. By the work of Farina [2], (0.1) does not possess any nontrivial solution since, by the Cwikel, Lieb and Rozenblum formula (see, for instance, Li and Yau [4]), the condition $u \in \dot{H}^2$ implies that u has finite Morse index (and we can apply the results in [2] for stable equations outside compact subsets of $\mathbb{R}^n$). In these short notes we propose a very direct path to prove this result using basic conformal geometry arguments and the underlying fourth order critical structure attached to (0.1). We assume that $n = 5, 6$ and prove that the following result holds true.

Theorem 0.1. *Suppose $n = 5, 6$. Equation (0.1) does not possess nontrivial solutions.*

The proof we propose is as follows. First we remark that if u solves (0.1), then u also solves the critical fourth order equation

$$\Delta^2 u = \frac{n}{n-4} |u|^{\frac{8}{n-4}} u - \frac{4n}{(n-4)^2} |u|^{\frac{2(6-n)}{n-4}} |\nabla u|^2 u \tag{0.2}$$

that we derive directly from (0.1) by letting Δ act on (0.1). Transposing (0.1) and (0.2) into S^n, using conformal arguments and basic regularity theory, we get that

$$u = \frac{a + f \circ \Phi_p^{-1}}{(1+|x|^2)^{\frac{n-4}{2}}} \ ,$$

where Φ_P is the stereographic projection of pole P, and $f \in C^{1,\theta}(S^n)$ vanishes at P. The "key" point is to prove that $a = 0$. Then, plugging u into the standard Pohozaev identity over balls $B_0(R)$ of large radii, letting $R \to +\infty$ and since $a = 0$, we get that $u = 0$.

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Proof. The first claim is that u satisfies (0.2) in $\mathbb{R}^n$ in the sense of distributions. Indeed, integrating by parts, for any $\varphi \in \dot{H}^2$,

$$\int_{\mathbb{R}^n} (\Delta u)(\Delta\varphi)dx = \int_{\mathbb{R}^n} |u|^{\frac{4}{n-4}} u(\Delta\varphi)dx = \frac{n}{n-4}\int_{\mathbb{R}^n} |u|^{\frac{4}{n-4}}(\nabla u \nabla\varphi)dx$$

and since $\dot{H}^2 \subset \dot{H}^{1,2n/(n-2)} \subset L^{2n/(n-4)}$, there holds that $|u|^{\frac{4}{n-4}}|\nabla u| \in L^{2n/(n+2)}$. Similarly,

$$\begin{aligned}\int_{\mathbb{R}^n} |u|^{\frac{4}{n-4}}(\nabla u \nabla\varphi)dx &= \int_{\mathbb{R}^n} |u|^{\frac{4}{n-4}}(\Delta u)\varphi dx - \frac{4}{n-4}\int_{\mathbb{R}^n} |u|^{\frac{2(6-n)}{n-4}} u|\nabla u|^2\varphi dx\\ &= \int_{\mathbb{R}^n} |u|^{\frac{8}{n-4}} u\varphi dx - \frac{4}{n-4}\int_{\mathbb{R}^n} |u|^{\frac{2(6-n)}{n-4}} u|\nabla u|^2\varphi dx\end{aligned}$$

and, here again, $|u|^{\frac{4}{n-4}}(\Delta u), |u|^{\frac{8}{n-4}}u, |u|^{\frac{2(6-n)}{n-4}}u|\nabla u|^2 \in L^{2n/(n+4)}$. As a remark, $\frac{n+4}{n-4} = \frac{2n}{n-4} - 1$ and $2^\sharp = \frac{2n}{n-4}$ is the critical Sobolev exponent for the embeddings $H^2 \subset L^p$. In particular, (0.2) is critical (while (0.1) was supercritical). Now we let $P \in S^n$ and Φ_P be the stereographic projection of pole P. Let also g_0 be the standard metric on S^n. Basic Riemannian geometry gives that

$$\left(\Phi_P^{-1}\right)^\star g_0 = \left(\frac{2}{1+|x|^2}\right)^2 \delta\ , \tag{0.3}$$

where δ is the Euclidean metric. Let $\varphi, \psi : \mathbb{R}^n \to \mathbb{R}$ be given by

$$\varphi(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-4}{2}} \quad\text{and}\quad \psi(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2}{2}}\ . \tag{0.4}$$

Then (0.3) rewrites as

$$\left(\Phi_P^{-1}\right)^\star g_0 = \varphi^{\frac{4}{n-4}}\delta = \psi^{\frac{4}{n-2}}\delta \tag{0.5}$$

and there also holds that $\psi = \varphi^{\frac{n-2}{n-4}}$. We define $\hat{u} : S^n\backslash\{P\} \to \mathbb{R}^n$ by

$$\hat{u} = \left(\frac{u}{\varphi}\right) \circ \Phi_P\ , \tag{0.6}$$

where φ is as in (0.4). Let P_g be the Paneitz [5] operator associated with a metric g. The operator P_g is conformally invariant in the sense that for any $u_0 > 0$ smooth, and any $w \in C^4$,

$$P_{u_0^{\frac{4}{n-4}}g} w = u_0^{-\frac{n+4}{n-4}} P_g(u_0 w)\ . \tag{0.7}$$

Let P_{g_0} be the Paneitz operator on S^n corresponding to $g = g_0$. Then P_{g_0} is given by

$$P_{g_0}u = \Delta_{g_0}^2 u + \frac{n^2-2n-4}{2}\Delta_{g_0}u + \frac{n(n-4)(n^2-4)}{16}u\ , \tag{0.8}$$

where $\Delta_{g_0} = -\text{div}_{g_0}\nabla$ is the Lapace-Beltrami operator. We also have that $P_\delta = \Delta^2$. By (0.2), (0.3), (0.5) and (0.7) we get that $\hat{u}$ as given in (0.6) satisfies that

$$P_{g_0}\hat{u} = \frac{n}{n-4}|\hat{u}|^{\frac{8}{n-4}}\hat{u} - F\hat{u} \tag{0.9}$$

in $S^n\backslash\{P\}$ in the sense of distributions, where

$$F = \frac{4n}{(n-4)^2}\left(|u|^{\frac{2(6-n)}{n-4}}|\nabla u|^2\varphi^{-\frac{8}{n-4}}\right)\circ\Phi_P\ , \tag{0.10}$$

and φ is as in (0.4). Since $\dot{H}^2 \subset \dot{H}^{1,2n/(n-2)} \subset L^{2n/(n-4)}$, there holds that

$$F \in L^{\frac{n}{4}}(S^n) \ . \tag{0.11}$$

Indeed, we can write by (0.5) that

$$\begin{aligned}\int_{S^n} |F|^{\frac{n}{4}} dv_{g_0} &= \int_{\mathbb{R}^n} \left|F \circ \Phi_P^{-1}\right|^{\frac{n}{4}} \varphi^{\frac{2n}{n-4}} dx \\ &= \int_{\mathbb{R}^n} |u|^{\frac{n(6-n)}{2(n-4)}} |\nabla u|^{\frac{n}{2}} dx \\ &\le \left(\int_{\mathbb{R}^n} |u|^{\frac{2n}{n-4}} dx\right)^{\frac{6-n}{4}} \left(\int_{\mathbb{R}^n} |\nabla u|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{4}}\end{aligned}$$

and (0.11) follows. Let $(u_\alpha)_\alpha$ be a sequence of smooth functions with compact support in $\mathbb{R}^n$ which converge to u in $\dot{H}^2$ and almost everywhere. By conformal invariance,

$$\int_{\mathbb{R}^n} \left(\Delta^2(u_\alpha - u_\beta)\right)(u_\alpha - u_\beta)dx = \int_{S^n} \left(P_{g_0}(\hat{u}_\alpha - \hat{u}_\beta)\right)(\hat{u}_\alpha - \hat{u}_\beta)dv_{g_0} \ ,$$

where the $\hat{u}_\alpha$ are given by the conformal rule (0.6), and since

$$\|u\|_{H^2} = \sqrt{\int_{S^n} (P_{g_0}u)udv_{g_0}}$$

is a norm on $H^2(S^n)$, we get that $(\hat{u}_\alpha)_\alpha$ is a Cauchy sequence in H^2. Since $\hat{u}_\alpha \to \hat{u}$ almost everywhere, we get that $\hat{u} \in H^2(S^n)$. As in Hebey and Robert [3], let $(\eta_s)_{s>0}$ be a family of smooth functions on S^n such that $0 \le \eta_s \le 1$, $\eta_s = 0$ in $B_P(s)$, $\eta_s = 1$ in $S^n \backslash B_P(2s)$, and

$$|\nabla \eta_s| \le \frac{C}{s} \ \text{ and } \ |\Delta_{g_0}\eta_s| \le \frac{C}{s^2} \tag{0.12}$$

for all $s > 0$, where $C > 0$ does not depend on s. Let also $\tilde{\eta}_s = \eta_s - 1$. Noting that by Hölder's inequalities,

$$\begin{aligned}&\lim_{s\to 0} \int_{S^n} (\Delta_{g_0}\hat{u})\left(\Delta_{g_0}(\tilde{\eta}_s v)\right) dv_{g_0} = 0 \ , \\ &\lim_{s\to 0} \int_{S^n} \left(\nabla_{g_0}\hat{u}\nabla_{g_0}(\tilde{\eta}_s v)\right) dv_{g_0} = 0 \ , \text{ and} \\ &\lim_{s\to 0} \int_{S^n} \hat{u}(\tilde{\eta}_s v)dv_{g_0} = 0\end{aligned} \tag{0.13}$$

for all $v \in H^2(S^n)$, and that

$$\lim_{s\to 0} \int_{S^n} |\hat{u}|^{\frac{8}{n-4}} \hat{u}(\tilde{\eta}_s v)dv_{g_0} = 0 \ \text{ and } \ \lim_{s\to 0} \int_{S^n} F\hat{u}(\tilde{\eta}_s v)dv_{g_0} = 0 \tag{0.14}$$

for all $v \in H^2(S^n)$ since $F\hat{u} \in L^{\frac{2n}{n+4}}$ by (0.11), we get by (0.13) and (0.14) that $\hat{u} \in H^2$ satisfies (0.9) in the sense of distributions in the whole of S^n. Then, by the regularity results in Djadli, Hebey and Ledoux [1], Lemma 2.1, we get that $\hat{u} \in L^p(S^n)$ for all $p \ge 1$. Now we exploit the second order equation satisfied by u. Let L_g be the conformal Laplacian given by

$$L_g u = \Delta_g u + \frac{n-2}{4(n-1)} S_g u \ ,$$

where S_g is the scalar curvature of g. Then L_g satisfies the conformal invariance rule that for any $u_0 > 0$ smooth, and any $w \in C^2$,

$$L_{u_0^{\frac{4}{n-2}}g} w = u_0^{-\frac{n+2}{n-2}} L_g(u_0 w) \ . \tag{0.15}$$

Let $\tilde{u} : S^n \backslash \{P\} \to \mathbb{R}$ be given by

$$\tilde{u} = \left(\frac{u}{\psi}\right) \circ \Phi_P \ , \tag{0.16}$$

where ψ is as in (0.4). There holds that

$$\hat{u} = (\varphi \circ \Phi_P)^{\frac{2}{n-4}} \tilde{u} \tag{0.17}$$

in $S^n \backslash \{P\}$, and if $Q = -P$, since

$$\Phi_P \circ \Phi_Q^{-1}(x) = \frac{x}{|x|^2} \ ,$$

we get that

$$(\varphi \circ \Phi_P)^{\frac{2}{n-4}} \left(\Phi_Q^{-1}(x)\right) = \frac{2|x|^2}{1+|x|^2} \tag{0.18}$$

for all $x \in \mathbb{R}^n \backslash \{0\}$.

Lemma 0.1. *There holds that* $\tilde{u} \in H^{2,\frac{n}{2}-\varepsilon}(S^n)$ *for all* $0 < \varepsilon \ll 1$, *and*

$$\Delta_{g_0}\tilde{u} + \frac{n(n-2)}{4}\tilde{u} = (\varphi \circ \Phi_P)^{-\frac{2}{n-4}} |\hat{u}|^{\frac{4}{n-4}} \hat{u} \tag{0.19}$$

in S^n *in the sense of distributions.*

Proof of Lemma 0.1. By (0.17) and (0.18) we have that in the chart $(S^n \backslash \{Q\}, \Phi_Q)$, from the viewpoint of integrability,

$$\tilde{u} \simeq \frac{1}{|x|^2}\hat{u}$$

at P, and since $\hat{u} \in L^p$ for all $p \geq 1$ according to what we proved above, we get that $\tilde{u} \in L^{\frac{n}{2}-\varepsilon}(S^n)$ for all $0 < \varepsilon \ll 1$. Let $v \in C^\infty(S^n)$ and $(\eta_s)_{s>0}$ be as above satisfying (0.12). By (0.1), (0.5), and (0.15) there holds that

$$\begin{aligned} \int_{S^n} \tilde{u}\left(L_{g_0}(\eta_s v)\right) dv_{g_0} &= \int_{\mathbb{R}^n} \left(\tilde{u} \circ \Phi_P^{-1}\right) L_{(\Phi_P^{-1})^\star g_0} \left((\eta_s v) \circ \Phi_P^{-1}\right) dv_{(\Phi_P^{-1})^\star g_0} \\ &= \int_{\mathbb{R}^n} u\Delta\left(\psi\left((\eta_s v) \circ \Phi_P^{-1}\right)\right) dx \\ &= \int_{\mathbb{R}^n} \left(|u|^{\frac{4}{n-4}} u\right) \psi\left((\eta_s v) \circ \Phi_P^{-1}\right) dx \end{aligned} \tag{0.20}$$

and we get that

$$\int_{S^n} \tilde{u}\left(L_{g_0}(\eta_s v)\right) dv_{g_0} = \int_{S^n} (\varphi \circ \Phi_P)^{-\frac{2}{n-4}} |\hat{u}|^{\frac{4}{n-4}} \hat{u}\eta_s v dv_{g_0} \tag{0.21}$$

There holds that $\eta_s v \to v$ in $L^p(S^n)$ for all $p \geq 1$, while

$$\Delta_{g_0}(\eta_s v) \to \Delta_{g_0} v$$

in $L^{\frac{n}{2}-\varepsilon}$ for all $0 < \varepsilon \ll 1$. Since $\hat{u} \in L^p$ for all $p \geq 1$, there also holds by (0.18) and Hölder's inequalities that

$$(\varphi \circ \Phi_P)^{-\frac{2}{n-4}} |\hat{u}|^{\frac{4}{n-4}} \hat{u} \in L^{\frac{n}{2}-\varepsilon}$$

for all $0 < \varepsilon \ll 1$. The conjugate exponent for $\frac{n}{2}$ is $\frac{n}{n-2}$ and we have that $\frac{n}{n-2} < \frac{n}{2}$. Hence,

$$\lim_{s\to 0}\int_{S^n} \tilde{u}\left(L_{g_0}(\eta_s v)\right) dv_{g_0} = \int_{S^n} \tilde{u}(L_{g_0}v)dv_{g_0}\ , \text{ and}$$
$$\lim_{s\to 0}\int_{S^n} (\varphi\circ\Phi_P)^{-\frac{2}{n-4}} |\hat{u}|^{\frac{4}{n-4}}\hat{u}\eta_s v dv_{g_0} = \int_{S^n} (\varphi\circ\Phi_P)^{-\frac{2}{n-4}} |\hat{u}|^{\frac{4}{n-4}}\hat{u} v dv_{g_0}\ ,$$

and we get by letting $s \to 0$ in (0.21) that $\tilde{u}$ satisfies (0.19) in S^n in the sense of distributions. Since, around P, in the chart $(S^n\backslash\{Q\},\Phi_Q)$,

$$\tilde{u} = \frac{1+|x|^2}{2|x|^2}\hat{u}$$

and $\hat{u} \in H^2$, there clearly holds that $\tilde{u} \in H^{1,p}(S^n)$ for $p > 1$ sufficiently close to 1. By regularity theory for second order elliptic equations it follows that

$$\tilde{u} \in H^{2,\frac{n}{2}-\varepsilon}$$

for all $0 < \varepsilon \ll 1$. This proves the lemma. □

Now we continue with the proof of Theorem 0.1. The same computations as in (0.20) give that

$$\int_{S^n} \tilde{u}L_{g_0}(\eta_s v)dv_{g_0} = \int_{S^n} (\psi\circ\Phi_P)^{\frac{8}{(n-2)(n-4)}} |\tilde{u}|^{\frac{4}{n-4}}\tilde{u}\eta_s v dv_{g_0} \tag{0.22}$$

for all $v \in C^\infty(S^n)$. By Lemma 0.1 there also holds that $\tilde{u} \in L^p$ for all $p \geq 1$. Hence we can let $s \to 0$ in (0.22) and we get that

$$\Delta_{g_0}\tilde{u} + \frac{n(n-2)}{4}\tilde{u} = (\psi\circ\Phi_P)^{\frac{8}{(n-2)(n-4)}} |\tilde{u}|^{\frac{4}{n-4}}\tilde{u} \tag{0.23}$$

in S^n in the sense of distributions. By regularity theory and (0.23), since $\psi\circ\Phi_P$ is bounded, we get that $\tilde{u} \in H^{2,p}$ for all $p \geq 1$, and it follows that

$$\tilde{u} \in C^{1,\theta}(S^n) \tag{0.24}$$

for all $0 < \theta < 1$. By (0.17), (0.18), and (0.24), we get that

$$\hat{u} \in C^{1,\theta}(S^n) \text{ and } \hat{u}(P) = 0 \tag{0.25}$$

for all $0 < \theta < 1$. This is the key assertion which makes that we can apply the Pohozaev identity. Given $\Omega \subset \mathbb{R}^n$ smooth and bounded, the Pohozaev identity for u is given by

$$\begin{aligned}&\frac{n-4}{2(n-2)}\int_{\partial\Omega} u^{\frac{2(n-2)}{n-4}}(x,\nu)d\sigma + \int_{\partial\Omega}(x,\nabla u)(\nu,\nabla u)d\sigma\\ &-\frac{1}{2}\int_{\partial\Omega}(x,\nu)|\nabla u|^2 d\sigma = \frac{n(n-4)}{2(n-2)}\int_\Omega u^{\frac{2(n-2)}{n-4}}dx - \frac{n-2}{2}\int_\Omega |\nabla u|^2 dx\ ,\end{aligned} \tag{0.26}$$

where ν is the unit outward normal derivative to $\partial\Omega$. We let $\Omega = B_0(R)$, $R \gg 1$. By (0.6),

$$u = \left(\hat{u}\circ\Phi_P^{-1}\right)\varphi\ .$$

Hence, by (0.25), we get that

$$u(x) = O\left(\frac{1}{|x|^{n-3}}\right) \text{ and } |\nabla u(x)| = O\left(\frac{1}{|x|^{n-2}}\right) \tag{0.27}$$

as $|x| \to +\infty$. Plugging (0.27) into (0.26), noting that by (0.1)

$$\int_\Omega |\nabla u|^2 dx = \int_\Omega u^{\frac{2(n-2)}{n-4}} dx + \int_{\partial\Omega} u(\nu, \nabla u) d\sigma \ ,$$

it follows that

$$\left(\frac{n(n-4)}{2(n-2)} - \frac{n-2}{2}\right) \int_{B_0(R)} |u|^{\frac{2(n-2)}{n-4}} dx = O\left(\frac{1}{R^{n-4}}\right) \tag{0.28}$$

for all $R \gg 1$. Letting $R \to +\infty$ in (0.28), we get that $u = 0$. This proves Theorem 0.1. □

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