AN INTRODUCTION TO FOURTH ORDER NONLINEAR WAVE EQUATIONS

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ABSTRACT. We discuss fourth order nonlinear wave equations in Euclidean space $\mathbb{R}^n$, n arbitrary. Given $m > 0$, the equations we consider write as

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + mu = f(u) \ ,$$

where $f(u)$ is a nonlinear term. We investigate well-posedness, blow-up in finite time, long time existence, and the existence of uniform bounds for global solutions of our equations. The text is intended to serve as basic notes and a possible source for an introductory graduate course on the subject.

CONTENTS

There has been an increasing activity in recent years on models involving nonlinear fourth-order partial differential equations. The very interesting book [49] by Peletier and Troy presents several such models which we can find in the physics literature. Fourth order equations have also been subject to an increasing activity in conformal geometry through the analysis of the Paneitz and Branson-Paneitz operators. We investigate in this paper fourth order wave equations in Euclidean space $\mathbb{R}^n$ which we write into the form

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + mu = f(u) \ , \tag{0.1}$$

where $m > 0$ is a positive real number, $\Delta = -\text{div}\nabla$ is the Laplace-Beltrami operator, and $f \in C^0(\mathbb{R}, \mathbb{R})$ is a continuous function such that $f(0) = 0$. The model case for (0.1) is given by the pure power nonlinearity $f(x) = \lambda|x|^{p-1}x$ where $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $p > 1$. At a first glance, (0.1) is a formal fourth-order extension of

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the classical Klein-Gordon equation. However it also inherits a Schrödinger structure which turns out to be of great help. Equations like (0.1) are also referred to as Bretherton's type equations or the beam equation. The original Bretherton equation, written down for $n = 1$ by Bretherton [6], arised in the study of weak interactions of dispersive waves. A similar equation for $n = 2$ was proposed in Love [43] for the motion of a clamped plate. The equation was discussed in Levine [38]. Recent developments in arbitrary dimension were established by Levandosky [35, 36], Levandosky and Strauss [37], Pausader [46], and Pausader and Strauss [47]. We also refer to Berger and Milewski [3], Berloff and Howard [4], Holm and Lynch [23], Lazer and McKenna [32], Lin [40], and McKenna and Walter [44, 45] for closely related references.

We address several questions in this paper such as well-posedness, blow-up in finite time, long time existence, and the existence of uniform bounds for global solutions of (0.1). As is well-known in control theory, the plate equation $\partial_t^2 u + \Delta^2 u = 0$ has a Schrödinger structure because of the decomposition $\partial_t^2 + \Delta^2 = (\partial_t + i\Delta)(\partial_t - i\Delta)$. Possible references in control theory, where the question of the plate equation is addressed, are Burq [7], Fu, Zhang, and Zuazua [15], Haraux [22], Jaffard [24], Lebeau [33, 34], Lions [41], Zhang [66], and Zuaza [67]. As is easily checked, (0.1) inherits the same structure. We exploit this structure through Strichartz estimates for the Schrödinger equation in Sections 1, 2, and 5 of the paper. The local theory for energy-subcritical Schrödinger equations was developed by Ginibre and Velo [18], and Kato [26]. A large and important part of the local theory for energy-critical Schrödinger equations was later on developed by Cazenave and Weissler [11, 12]. We transpose and adapt several of their arguments to (0.1) in various places of the paper. Hyperbolic tools developed for Klein-Gordon equations, in particular by Cazenave [8], are used instead in Sections 3 and 4. As a remark, uniform bounds like the ones we get in Section 5 were originally proved in [8] for Klein-Gordon equations. We proceed here with a slightly different approach using the Schrödinger structure of the equation. Stability, following the approach in Struwe [60], is proved in Section 8.

1. Local Existence

We are concerned in this section with proving local existence of strong solutions of (0.1) with given Cauchy data. We assume in what follows that $f \in C^0(\mathbb{R}, \mathbb{R})$ satisfies that $f(0) = 0$ and that there exists $C > 0$ such that

$$|f(y) - f(x)| \leq C \left(1 + |x|^{p-1} + |y|^{p-1}\right) |y - x| \tag{1.1}$$

for all $x, y \in \mathbb{R}$, where $p > 1$ is arbitrary if $1 \leq n \leq 4$, $1 < p \leq 2^\sharp - 1$ if $n \geq 5$, and $2^\sharp = 2n/(n-4)$. Let $H^2 = H^{2,2}(\mathbb{R}^n)$ be the Sobolev space of functions in L^2 with two derivatives in L^2, and let $\|\cdot\|_{H^2}$ be the norm on H^2 given by

$$\|u\|_{H^2}^2 = \int_{\mathbb{R}^n} \left((\Delta u)^2 + mu^2\right) dx\ . \tag{1.2}$$

The exponent $2^\sharp$ is the critical Sobolev exponent for the embedding of H^2 into Lebesgue spaces. For f as in (1.1), and for $(u, v) \in H^2 \times L^2$, we define the total energy $E(u, v)$ and the kinetic energy $E_0(u, v)$ by

$$E_0(u, v) = \frac{1}{2} \left(\|u\|_{H^2}^2 + \|v\|_{L^2}^2\right) \text{ and } E(u, v) = E_0(u, v) - \int_{\mathbb{R}^n} F(u) dx\ , \tag{1.3}$$

where F is the primitive of f given by $F(x) = \int_0^x f(t)dt$ for all $x \in \mathbb{R}$, and $\|\cdot\|_{L^q}$ is the L^q-norm in $\mathbb{R}^n$ for $q \geq 1$. Equation (0.1) is energy-critical when $p = 2^\sharp - 1$ and $n \geq 5$. Given $u_0 \in H^2$ and $u_1 \in L^2$, we say that u is a solution of (0.1) in $[0, T)$ with Cauchy data u_0 and u_1 if

$$\begin{aligned}&u \in C^0([0,T), H^2) \cap C^1([0,T), L^2) \cap C^2([0,T), H^{-2}) ,\\ &\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + mu = f(u) \text{ in } C^0([0,T), H^{-2}) , \text{ and}\\ &u_{|t=0} = u_0 , \ u_{t|t=0} = u_1 ,\end{aligned} \tag{1.4}$$

where H^{-2} stands for the topological dual space of H^2. By extension, if I is an interval such that $0 \in I$, we say that u solves (0.1) with Cauchy data u_0 and u_1 if (1.4) holds with I in place of $[0, T)$. Such solutions are referred to as strong solutions. We prove in Theorem 1.1 below that we do have local existence for strong solutions of (0.1) with general nonlinear terms f as in (1.1), including the energy-critical case. To do this we derive Strichartz type estimates for (0.1) from the Schrödinger structure of the equation and Strichartz's estimates for the Schrödinger equation. In what follows we say that a pair (q, r) is Schrödinger admissible, for short S-admissible, if

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2} \tag{1.5}$$

and r is such that $2 \leq r \leq +\infty$ if $n = 1$, $2 \leq r < +\infty$ if $n = 2$, and $2 \leq r \leq 2^\star$ if $n \geq 3$, where $2^\star = \frac{2n}{n-2}$. For $2 \leq q \leq +\infty$, we say that a pair (q, r) is beam admissible, for short B-admissible, if $2 \leq r \leq +\infty$ when $n = 1, 2, 3$, $2 \leq r < +\infty$ when $n = 4$, and

$$\frac{2}{q} + \frac{n}{r} = \frac{n-4}{2} \tag{1.6}$$

with $0 < r < +\infty$ when $n \geq 5$. If (q, r) is S-admissible and $2r < n$, then $(q, r^\sharp)$ is B-admissible for $r^\sharp = \frac{nr}{n-2r}$. Note that $s = r^\sharp$ is the critical Sobolev exponent for the embedding of $H^{2,r}$ into L^s, where $H^{2,r}$ stands for the Sobolev space of functions in L^r with two derivatives in L^r. More generally, given $s \in \mathbb{R}$ and $p \geq 1$, we let $H^{s,p} = H^{s,p}(\mathbb{R}^n)$ be the usual fractional Sobolev spaces in $\mathbb{R}^n$. Following standard notations we let also $H^s = H^{s,2}$, and for $q \geq 1$ we let q' be the conjugate of q. Local in time Strichartz type estimates for (0.1) are as follows. Global in time Strichartz estimates are proved in Pausader [46].

Lemma 1.1. *Let $I \subset \mathbb{R}$ be a bounded interval such that $0 \in I$, $u_0 \in H^2$, $u_1 \in L^2$, and $k \in C^0(I, H^{-2}) \cap L^{a'}(I, L^{b'})$ for some S-admissible pair (a, b). There exists a unique $u \in C^0(I, H^2) \cap C^1(I, L^2) \cap C^2(I, H^{-2})$ which solves the linear equation*

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u = k \tag{1.7}$$

in $C^0(I, H^{-2})$ with Cauchy data $u_{|t=0} = u_0$ and $u_{t|t=0} = u_1$. Moreover it holds that $u \in L^q(I, L^r)$ for any B-admissible pair (q, r), and that

$$\begin{aligned}&\|u\|_{C^0(I,H^2)} + \|u_t\|_{C^0(I,L^2)} + \|u\|_{L^q(I,L^r)}\\ &\leq C\left(1 + |I|^{3/2}\right)\left(\sqrt{E_0(u_0,u_1)} + \|k\|_{L^{a'}(I,L^{b'})}\right) ,\end{aligned} \tag{1.8}$$

where $|I|$ is the length of I, E_0 is as in (1.3), and $C \geq 1$ does not depend on u_0, u_1, k, and I.

Proof. We let v solve (1.7) in $C^0(I, H^{-4})$ with Cauchy data $v_{|t=0} = 0$ and $v_{t|t=0} = 0$. We let also w solve (1.7) in $C^0(I, H^{-2})$ when $k \equiv 0$ with Cauchy data $w_{|t=0} = u_0$ and $w_{t|t=0} = u_1$. By standard Fourier analysis, v and w exist. Also we obtain that $v \in C^0(I, L^2) \cap C^1(I, H^{-2}) \cap C^2(I, H^{-4})$ and $w \in C^0(I, H^2) \cap C^1(I, L^2) \cap C^2(I, H^{-2})$. Let $\tilde{v} = -iv_t + \Delta v$ and $\tilde{w} = -iw_t + \Delta w$. We consider the linear Schrödinger equation

$$iu_t + \Delta u = k \ . \tag{1.9}$$

As is easily checked, $\tilde{v}$ solves (1.9) in $C^0(I, H^{-4})$ with Cauchy data $\tilde{v}_{|t=0} = 0$, and $\tilde{w}$ solves (1.9) in $C^0(I, H^{-2})$ when $k \equiv 0$ with Cauchy data $\tilde{w}_{|t=0} = -iu_1 + \Delta u_0$. We may then apply the standard Strichartz estimates for the Schrödinger equation, as stated for instance in Cazenave [9], to $\tilde{v}$ and $\tilde{w}$. We refer also to Keel and Tao [28]. The Strichartz estimates for $\tilde{v}$ give that $\tilde{v} \in C^0(I, L^2) \cap L^q(I, L^s)$ for any S-admissible pair (q, s), and that the L^qL^s-norm of $\tilde{v}$ is controled by the $L^{a'}L^{b'}$-norm of k. This includes the choice of (q, s) given by $q = +\infty$ and $s = 2$. In particular, it follows that $v \in C^0(I, H^2) \cap C^1(I, L^2) \cap C^2(I, H^{-2})$, and by considering the real and imaginary parts of $\tilde{v}$ we also get that for any S-admissible pair (q, s),

$$\|\Delta v\|_{C^0(I,L^2)} + \|v_t\|_{C^0(I,L^2)} + \|\Delta v\|_{L^q(I,L^s)} \le C\|k\|_{L^{a'}(I,L^{b'})} \ , \tag{1.10}$$

where $C > 0$, independent of I, depends only on n, (a, b), and (q, s). As a remark this implies that v solves (1.7) in $C^0(I, H^{-2})$ and not only in $C^0(I, H^{-4})$. By the control on the L^2-norm of v_t in (1.10), and since $\frac{d}{dt}\|v\|_{L^2}^2 \le 2\|v_t\|_{L^2}\|v\|_{L^2}$, we can write that

$$\begin{aligned} &\|v\|_{C^0(I,H^2)} + \|v_t\|_{C^0(I,L^2)} + \|\Delta v\|_{L^q(I,L^s)} \\ &\le C\left(1 + |I|\right)\|k\|_{L^{a'}(I,L^{b'})} \ , \end{aligned} \tag{1.11}$$

where $C > 0$, independent of I, depends only on n, m, (a, b), and (q, s). Let (q, r) be a B-admissible pair as in the statement of Lemma 1.1. When $n \le 4$, and since $r \ge 2$, we can write by the Sobolev embedding theorem for H^2, and by the inclusion $H^2 \subset H^{s,2}$ for $s \le 2$ and the Sobolev embedding theorem for fractional Sobolev spaces when $n = 4$, that

$$\|v\|_{L^q(I,L^r)} \le C|I|^{1/q}\|v\|_{C^0(I,H^2)} \le C\left(1 + |I|^{1/2}\right)\|v\|_{C^0(I,H^2)} \ , \tag{1.12}$$

where $C > 0$ depends only on n and (q, r). When $n \ge 5$, we let s be given by $s = nr/(n+2r)$. Then (q, s) is S-admissible and $s^\sharp = r$. From Adams and Fournier [1], and Stein [56] the $L^{s^\sharp}$-norm of a smooth function u with compact support is controled by a dimensional constant times the L^s-norm of its Laplacian Δu when $1 \le s < n/2$. With our choice of s, and by approximation, we may then write that

$$\|v\|_{L^q(I,L^r)} \le C\|\Delta v\|_{L^q(I,L^s)} \ , \tag{1.13}$$

where $C > 0$ depends only on n and (q, r). Combining (1.11), (1.12), and (1.13) we get that

$$\|v\|_{C^0(I,H^2)} + \|v_t\|_{C^0(I,L^2)} + \|v\|_{L^q(I,L^r)} \le C\left(1 + |I|^{3/2}\right)\|k\|_{L^{a'}(I,L^{b'})} \ , \tag{1.14}$$

where $C > 0$, independent of I, depends only on n, m, (a,b), and (q,r). Similarly, the Strichartz's estimates for $\tilde{w}$ give that

$$\begin{aligned}
&\|w\|_{C^0(I,H^2)} + \|w_t\|_{C^0(I,L^2)} + \|w\|_{L^q(I,L^r)} \\
&\le C\left(1+|I|^{3/2}\right)\left(\|u_1\|_{L^2} + \|u_0\|_{L^2} + \|\Delta u_0\|_{L^2}\right) \\
&\le C\left(1+|I|^{3/2}\right)\sqrt{E_0(u_0,u_1)}\ ,
\end{aligned} \tag{1.15}$$

where $C \ge 1$, independent of I, depends only on n, m, and (q,r). By (1.14) and (1.15), letting $u = v+w$, we get a solution of (1.7) in $C^0(I,H^{-2})$ with Cauchy data $u_{|t=0} = u_0$ and $u_{t|t=0} = u_1$ which satisfies (1.8) for any B-admissible pair (q,r). Uniqueness of u follows from the remark that if u_1 and u_2 are two such solutions, then $\tilde{u} = u_2 - u_1$ solves (1.7) with $k = 0$ and Cauchy data $\tilde{u}_{|t=0} = 0$ and $\tilde{u}_{t|t=0} = 0$ so that $\tilde{u} = 0$. This proves Lemma 1.1. □

As a remark, the proof of Lemma 1.1 also gives that $u_t \in L^q(I,L^s)$ for any S-admissible pair (q,s). Since $2 \le s \le 2^\sharp$ for such pairs, and $u \in C^0(I,H^2)$, we also get from the Sobolev embedding theorem that $u \in L^q(I,L^s)$. In Theorem 1.1 below we establish local existence of strong solutions for (0.1). Complementary corollaries and remarks on the theorem, like well-posedness and a caracterisation of the explosion in terms of norms and mixed-norms, are discussed in Section 2.

Theorem 1.1. *Let f satisfy (1.1), $u_0 \in H^2$, and $u_1 \in L^2$. There exists a unique solution u of (0.1) with Cauchy data u_0, u_1 defined on a maximal time interval $[0,T)$. Furthemore, $E(u,u_t) = E(u_0,u_1)$ for all $t \in [0,T)$, where E is the total energy as in (1.3), $u = u(t)$, and $u_t = u_t(t)$.*

In order to prove the theorem we let $h \in C^0(\mathbb{R},\mathbb{R})$ be given by $h(u) = f(u) - mu$, and rewrite equation (0.1) into the form

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u = h(u)\ . \tag{1.16}$$

As is easily checked, h also satisfies (1.1). We let $\eta : \mathbb{R} \to \mathbb{R}$ be smooth, with compact support, and such that $\eta = 1$ in $[-1,1]$. We define $h_1 = \eta h$ and $h_2 = (1-\eta)h$. Then, by (1.1),

$$\begin{aligned}
&h = h_1 + h_2\ ,\ h_1 \text{ is Lipschitz}\ , \text{ and} \\
&|h_2(y) - h_2(x)| \le C\left(|x|^{p-1} + |y|^{p-1}\right)|y-x|
\end{aligned} \tag{1.17}$$

for all $x,y \in \mathbb{R}$, where p is as in (1.1), and $C > 0$ is independent of x and y. We also have that $h_1(0) = h_2(0) = 0$. For $T > 0$ we define

$$\mathcal{H}_T = C^0([0,T],H^2) \cap C^1([0,T],L^2)\ \ ,\ \ \tilde{\mathcal{H}}_T = \mathcal{H}_T \cap C^2([0,T],H^{-2}) \tag{1.18}$$

and, when $n \ge 5$, we let also

$$\hat{\mathcal{H}}_T = \mathcal{H}_T \cap L^{q_n}([0,T],L^{r_n})\ , \tag{1.19}$$

where (q_n,r_n) is B-admissible and given by $q_n = 2(2^\sharp - 1)$, $r_n = 2^\sharp(n+4)/(n+2)$. For the reader's convenience, we divide the proof of Theorem 1.1 into three parts where we respectively prove existence, conservation of the energy, and uniqueness. We start with existence. In the process we distinguish the subcritical case from the critical case because of the different nature of the arguments we use in both cases which, we refer to the remarks after the proof, provide different informations on the lifespan and the energy of the solution.

Proof of Theorem 1.1 – Existence. (i) We assume either that $n \leq 4$ or that $n \geq 5$ and $p < 2^\sharp - 1$ in (1.1). We prove that for any $u_0 \in H^2$ and any $u_1 \in L^2$ there exists $T > 0$ such that (0.1) possesses a solution u with Cauchy data u_0, u_1 defined on the time interval $[0,T)$. In order to do this we let h, h_1, and h_2 be as in (1.17), and we consider (0.1) when written under the form of equation (1.16). When $n \geq 5$ we let (q,r) be the B-admissible pair given by $q = q_n$ and $r = r_n$, where q_n and r_n are as in (1.19), and for $T > 0$ we let $\mathcal{H}$ be the Banach space given by $\mathcal{H} = \mathcal{H}_T$ if $n \leq 4$, and $\mathcal{H} = \hat{\mathcal{H}}_T$ if $n \geq 5$, where $\mathcal{H}_T$ and $\hat{\mathcal{H}}_T$ are as in (1.18) and (1.19). Let $u \in \mathcal{H}$ be arbitrary. Since h_1 is Lipschitz and $h_1(0) = 0$, we can write that $h_1(u) \in C^0(I, L^2)$ while , by (1.17), $h_2(u) \in C^0(I, L^{2n/(n+4)})$. In particular, both $h_1(u)$ and $h_2(u)$ are in $C^0([0,T], H^{-2})$. We also get that $h_1(u) \in L^1([0,T], L^2)$ so that $h_1(u) \in L^{a'}([0,T], L^{b'})$ for (a,b) the S-admissible pair given by $a = +\infty$ and $b = 2$. Similarly, by the Sobolev embedding theorem for H^2 when $n \leq 3$, and by the inclusion $H^2 \subset H^{s,2}$ for $s \leq 2$ and the Sobolev embedding theorem for fractional Sobolev spaces when $n = 4$, we have that $h_2(u) \in L^1([0,T], L^2)$ when $n \leq 4$. Moreover, we can write that

$$\begin{aligned} &\|h_1(u)\|_{L^1([0,T],L^2)} \leq CT\|u\|_{C^0([0,T],L^2)} \text{ for all } n \ , \\ &\|h_2(u)\|_{L^1([0,T],L^2)} \leq CT\|u\|^p_{C^0([0,T],H^2)} \text{ when } n \leq 4 \ , \end{aligned} \tag{1.20}$$

where $C > 0$ depends only on f, n and m. Without loss of generality, we may assume in what follows that $p \geq (2^\sharp - 1)n/(n+2)$ in (1.1) when $n \geq 5$. Then $2n/(n+2) \leq r/p \leq 2$ for the above choice of r. In particular, assuming that $n \geq 5$, we get that there exists a S-admissible pair (c,d) such that $pd' = r$. Since $p < 2^\sharp - 1$, we have that $pc' < q$ and we let $\delta > 0$ be such that

$$\frac{1}{pc'} = \frac{1}{q} + \frac{\delta}{p} \ . \tag{1.21}$$

By (1.17) and Hölder's inequality we then get that $h_2(u) \in L^{c'}([0,T], L^{d'})$ and that

$$\|h_2(u)\|_{L^{c'}([0,T],L^{d'})} \leq C\|u\|^p_{L^{pc'}([0,T],L^{pd'})} \leq CT^\delta \|u\|^p_{L^q([0,T],L^r)} \ , \tag{1.22}$$

where $C > 0$ depends only on f, m, and n. We fix $u_0 \in H^2$ and $u_1 \in L^2$. Also we let (c,d) be as above when $n \geq 5$, and let $(c,d) = (+\infty, 2)$ when $n \leq 4$. For $u \in \mathcal{H}$ we consider the linear equation (1.7) with $k = h(u)$ and Cauchy data u_0, u_1. The solution $v = \chi(u)$ of this linear problem writes as the sum of the solution of (1.7) with $k = 0$ and Cauchy data (u_0, u_1), the solution of (1.7) with $k = h_1(u)$ and Cauchy data $(0,0)$, and the solution of (1.7) with $k = h_2(u)$ and Cauchy data $(0,0)$. By the linear theory in Lemma 1.1, by (1.20), and by (1.22), we can write that $v = \chi(u)$ belongs to $\mathcal{H}$ and that

$$\begin{aligned} \|v\|_{\mathcal{H}} &\leq C_T \left(\sqrt{E_0(u_0,u_1)} + \|h_1(u)\|_{L^1([0,T],L^2)} + \|h_2(u)\|_{L^{c'}([0,T],L^{d'})} \right) \\ &\leq C_T \left(\sqrt{E_0(u_0,u_1)} + T\|u\|_{\mathcal{H}} + T^\delta \|u\|^p_{\mathcal{H}} \right) \ , \end{aligned} \tag{1.23}$$

where $\delta > 0$ equals 1 if $n \leq 4$, and $C_T = C\left(1 + T^{3/2}\right)$ for some $C > 0$ depending only on f, n, and m. In particular, we defined a map $\chi : \mathcal{H} \to \mathcal{H}$ and by (1.23), we see that for any $M > 2C\sqrt{E_0(u_0,u_1)}$, there exists $T > 0$ sufficiently small depending only on f, n, M, $E_0(u_0,u_1)$, and m, such that χ preserves the close ball $B'_0(M)$ of center 0 and radius M in $\mathcal{H}$, where $C > 0$ is as in (1.23). For instance, by noting that $\delta \le 1$, we get that $\chi : B_0'(M) \to B_0'(M)$ for $T \in (0,1]$ such that

$$T \le \left(\frac{M - 2C\sqrt{E_0(u_0,u_1)}}{2C(M+M^p)}\right)^{1/\delta} .$$

By (1.17) there exists $C > 0$ depending only on f, m, and n such that

$$\begin{aligned} &\|h_2(v) - h_2(u)\|_{L^{c'}([0,T],L^{d'})} \\ &\le C\left(\|u\|^{p-1}_{L^{pc'}([0,T],L^{pd'})} + \|v\|^{p-1}_{L^{pc'}([0,T],L^{pd'})}\right)\|v-u\|_{L^{pc'}([0,T],L^{pd'})} \end{aligned} \tag{1.24}$$

for all $u, v \in \mathcal{H}$. By (1.24), Hölder's inequality, and the linear theory developed in Lemma 1.1, we can then write that for $u, v \in B_0'(M)$,

$$\begin{aligned} &\|\chi(v) - \chi(u)\|_{\mathcal{H}} \\ &\le C\left(\|h_1(v) - h_1(u)\|_{L^1([0,T],L^2)} + \|h_2(v) - h_2(u)\|_{L^{c'}([0,T],L^{d'})}\right) \\ &\le C\Big(T\|v-u\|_{\mathcal{H}} \\ &\qquad + T^\delta\left(\|u\|^{p-1}_{L^q([0,T],L^r)} + \|v\|^{p-1}_{L^q([0,T],L^r)}\right)\|v-u\|_{L^q([0,T],L^r)}\Big) \\ &\le C\left(T + 2T^\delta M^{p-1}\right)\|v-u\|_{\mathcal{H}} \end{aligned} \tag{1.25}$$

where $\delta > 0$ is as in (1.21) if $n \ge 5$, $\delta = 1$ if $n \le 4$, and $C > 0$ depends only on f, n, and m. In particular, for $M > 2C\sqrt{E_0(u_0,u_1)}$ and $T > 0$ sufficiently small depending only on f, n, m, M, and $E_0(u_0,u_1)$, the map $\chi : B_0'(M) \to B_0'(M)$ acts as a contraction. By the Banach fixed point theorem we then get that χ has a fixed point in $\mathcal{H}$. This proves the above claim that when $n \le 4$, or when $n \ge 5$ and $p < 2^\sharp - 1$ in (1.1), then, for any $u_0 \in H^2$ and any $u_1 \in L^2$, there exists $T > 0$ such that (0.1) possesses a solution u with Cauchy data u_0, u_1 defined on the time interval $[0,T)$.

(ii) We assume that $n \ge 5$ and that $p = 2^\sharp - 1$ in (1.1). We prove that for any $u_0 \in H^2$ and any $u_1 \in L^2$ there exists $T > 0$ such that (0.1) possesses a solution u with Cauchy data u_0, u_1 defined on the time interval $[0,T)$. As above we consider (0.1) when written under the form of equation (1.16). We let (q,r) and (a,b) be the B-admissible and S-admissible pairs given by $q = q_n$ and $r = r_n$, where q_n, r_n are as in (1.19), and by $a = 2$ and $b = 2n/(n-2)$. In particular, $r = b'(2^\sharp - 1)$. For $T > 0$, we let $\mathcal{H}$ be the Banach space $\mathcal{H} = \hat{\mathcal{H}}_T$, where $\hat{\mathcal{H}}_T$ is as in (1.19). For $u \in \mathcal{H}$, as in (i) above, we easily get that $h_1(u) \in L^1([0,T],L^2)$, $h_2(u) \in L^{a'}([0,T],L^{b'})$, and that

$$\begin{aligned} &\|h_1(u)\|_{L^1([0,T],L^2)} \le CT\|u\|_{C^0([0,T],L^2)} , \\ &\|h_2(u)\|_{L^{a'}([0,T],L^{b'})} \le C\|u\|^p_{L^{pa'}([0,T],L^{pb'})} \le C\|u\|^p_{L^q([0,T],L^r)} , \end{aligned} \tag{1.26}$$

where $p = 2^\sharp - 1$, and $C > 0$ depends only on f, n and m. Let $\mathcal{U}$ be the solution given by Lemma 1.1 of the linear equation (1.7) with $k = 0$ and Cauchy data $\mathcal{U}_{|t=0} = u_0$, $\mathcal{U}_{t|t=0} = u_1$. In particular,

$$\frac{\partial^2\mathcal{U}}{\partial t^2} + \Delta^2\mathcal{U} = 0 \tag{1.27}$$

in $C^0(\mathbb{R}, H^{-2})$. For $u \in \mathcal{H}$, we let also $v = \chi(u)$ be the solution of (1.7) with $k = h(u)$ and Cauchy data u_0, u_1. Then v writes as the sum of $\mathcal{U}$, the solution of (1.7) with $k = h_1(u)$ and Cauchy data $(0,0)$, and the solution of (1.7) with

$k = h_2(u)$ and Cauchy data $(0, 0)$. By the linear theory in Lemma 1.1 we get that $\chi : \mathcal{H} \to \mathcal{H}$, and when taking into consideration the estimates (1.26) we can write that

$$\|v\|_{L^q([0,T],L^r)} \leq \|\mathcal{U}\|_{L^q([0,T],L^r)} + C_T \left(T\|u\|_{\mathcal{H}} + \|u\|^p_{L^q([0,T],L^r)} \right) \tag{1.28}$$

and that

$$\|v\|_{\mathcal{H}} \leq C_T \left(\sqrt{E_0(u_0, u_1)} + T\|u\|_{\mathcal{H}} + \|u\|^p_{L^q([0,T],L^r)} \right) \tag{1.29}$$

where $p = 2^\sharp - 1$, $C_T = C(1 + T^{3/2})$, and $C > 0$ depends only f, n, and m. Given $\delta > 0$ arbitrary, we let $T_\delta \in (0, 1)$ be such that $\|\mathcal{U}\|_{L^q([0,T],L^r)} \leq \delta$ for all $T \in (0, T_\delta)$, and for $s > 0$ and $M > 0$ we define the closed set $Y^s_{T,M} \subset \mathcal{H}$ by

$$Y^s_{T,M} = \left\{ u \in \mathcal{H} \text{ s.t. } \|u\|_{L^q([0,T],L^r)} \leq s \text{ and } \|u\|_{\mathcal{H}} \leq M \right\} . \tag{1.30}$$

By (1.28) and (1.29) we easily get that for $\delta, s > 0$ sufficiently small, and $M > 0$ sufficiently large, there exists $T \in (0, T_\delta)$ such that χ preserves $Y^s_{T,M}$. For instance, by choosing $M > 2C\sqrt{E_0(u_0, u_1)}$ and $\delta, s > 0$ sufficiently small such that

$$2Cs^p \leq s/4 \ , \ 2Cs^p + s/4 < M - 2C\sqrt{E_0(u_0, u_1)} \ , \text{ and } \delta \leq s/2 \ , \tag{1.31}$$

we get that $\chi : Y^s_{T,M} \to Y^s_{T,M}$ if $T \in (0, T_\delta]$ is such that $2CTM \leq s/4$. Independently, by an inequality like (1.24) with (a, b) in place of (c, d), by Hölder's inequality, and the linear theory developed in Lemma 1.1, we can write that for $u, v \in Y^s_{T,M}$,

$$\begin{aligned}
&\|\chi(v) - \chi(u)\|_{\mathcal{H}} \\
&\leq C \left(\|h_1(v) - h_1(u)\|_{L^1([0,T],L^2)} + \|h_2(v) - h_2(u)\|_{L^{a'}([0,T],L^{b'})} \right) \\
&\leq C\Big(T\|v - u\|_{\mathcal{H}} + \left(\|u\|^{p-1}_{L^q([0,T],L^r)} + \|v\|^{p-1}_{L^q([0,T],L^r)} \right) \|v - u\|_{L^q([0,T],L^r)} \Big) \\
&\leq C \left(T + 2s^{p-1} \right) \|v - u\|_{\mathcal{H}} \ ,
\end{aligned} \tag{1.32}$$

where $C > 0$ depends only on f, n, and m. In particular, for $s, T > 0$ sufficiently small, the map $\chi : Y^s_{T,M} \to Y^s_{T,M}$ is a contraction. By the Banach fixed point theorem we then get that χ has a fixed point in $\mathcal{H}$. This proves the above claim that when $n \geq 5$ and $p = 2^\sharp - 1$ in (1.1), then, for any $u_0 \in H^2$ and any $u_1 \in L^2$, there exists $T > 0$ such that (0.1) possesses a solution u with Cauchy data u_0, u_1 defined on the time interval $[0, T)$. □

As a remark, the above proof provides uniqueness of the solution in $\tilde{\mathcal{H}}_T$-spaces when $n \leq 4$, as predicted by Theorem 1.1, and in $\tilde{\mathcal{H}}_T \cap L^{q_n}([0, T], L^{r_n})$-spaces when $n \geq 5$, where $\tilde{\mathcal{H}}_T$, q_n, and r_n are as in (1.18) and (1.19). From now on we let $T^\star = T^\star(u_0, u_1)$ stand for the maximal time of existence of the solution of (0.1) with Cauchy data u_0 and u_1. Assuming either that $n \leq 4$, or that $n \geq 5$ and $p < 2^\sharp - 1$ in (1.1), the time for the contraction in point (i) of the above proof can be chosen such that it depends only on f, n, m, and $E_0(u_0, u_1)$. It follows that when $n \leq 4$, or when $n \geq 5$ and $p < 2^\sharp - 1$ in (1.1), then $E_0(u, u_t) \to +\infty$ as $t \to T^\star$ if $T^\star < +\infty$. By taking $M = C(1 + \sqrt{E_0(u_0, u_1)})$, with $C \gg 1$, and assuming that $p \geq (2^\sharp - 1)n/(n + 2)$ in (1.1), we also get that when $n \leq 4$, or when $n \geq 5$ and $p < 2^\sharp - 1$ in (1.1), then there exists $K = K(f, n, m) \in (0, 1)$, depending only on

f, n, and m, such that for any $u_0 \in H^2$ and $u_1 \in L^2$,

$$T^\star \geq \frac{K}{\left(1+\sqrt{E_0(u_0,u_1)}\right)^{\frac{p-1}{\delta}}}\ , \tag{1.33}$$

where $\delta = 1$ if $n \leq 4$ and $\delta > 0$ is as in (1.21) if $n \geq 5$. Now let us assume that $n \geq 5$ and $p = 2^\sharp - 1$ in (1.1). Then we cannot assert anymore from (ii) in the above proof that $E_0(u, u_t) \to +\infty$ as $t \to T^\star$ if $T^\star < +\infty$. We prove instead, in Section 2, that $\|u\|_{L^{q_n}([0,T],L^{r_n})} \to +\infty$ as $T \to T^\star$. On the other hand, for an analogue of (1.33) in the critical case, it follows from point (ii) in the above proof that there exists $\delta > 0$ small such that for any $u_0 \in H^2$ and $u_1 \in L^2$, if $\|\mathcal{U}\|_{L^{q_n}([0,T],L^{r_n})} < \delta$ for some $T \in (0, 1)$, then $T^\star = T^\star(u_0, u_1)$ is such that

$$T^\star \geq \frac{\delta T}{\sqrt{1+E_0(u_0,u_1)}}\ , \tag{1.34}$$

where $\mathcal{U}$ is the solution of the linear equation (1.27) with Cauchy data u_0 and u_1. Independently, it can be noted that in all cases, by the linear theory in Lemma 1.1, see also the remark after the proof of Lemma 1.1, we do get that $u \in L^q_{loc}([0,T^\star),L^r)$ and $u_t \in L^a_{loc}([0,T^\star),L^b)$ for all B-admissible pairs (q,r), and all S-admissible pairs (a,b). Now we prove that the conservation of the energy in Theorem 1.1 holds true.

Proof of Theorem 1.1 – Conservation of the energy. We prove in what follows that if $u \in \tilde{\mathcal{H}}_T \cap L^{q_n}([0,T],L^{r_n})$ solves (0.1) with Cauchy data u_0 and u_1, where $\tilde{\mathcal{H}}_T$, q_n, and r_n are as in (1.18) and (1.19), then, for any $t \in [0,T]$, $E(u,u_t) = E(u_0,u_1)$, where E is the total energy as in (1.3), $u = u(t)$, and $u_t = u_t(t)$. For this purpose we consider (0.1) when written under the form of equation (1.16), and let E_0', E' be like E_0 and E in (1.3) when $m = 0$ and $F = H$ is the primitive of h given by $H(x) = \int_0^x h(t)dt$ for all $x \in \mathbb{R}$. The proof of the conservation of the energy in Theorem 1.1 reduces to proving that $E'(u,u_t) = E'(u_0,u_1)$ in $[0,T]$ if $u \in \mathcal{H}_T$ solves (1.16) with Cauchy data u_0, u_1. For $\varepsilon > 0$ we let $J_\varepsilon = (I + \varepsilon\Delta)^{-1}$. Then, see for instance Cazenave [9], for any s, J_ε is a contraction in H^s , $J_\varepsilon \in \mathcal{L}(H^s, H^{s+2})$ with a norm of the order of ε^{-1} when $\varepsilon > 0$ is small, and

$$\|J_\varepsilon v - v\|_{H^s} \leq \varepsilon\|\Delta v\|_{H^s} \tag{1.35}$$

for all $v \in H^{s+2}$. In particular, by (1.35), the density of H^{s+2} into H^s, and the contraction property of J_ε in H^s, we easily get that $J_\varepsilon v \to v$ in H^s for all $v \in H^s$ as $\varepsilon \to 0$. We also have that for any $q > 1$ and any $\varepsilon > 0$, J_ε is a contraction of L^q and $J_\varepsilon v \to v$ in L^q for all $v \in L^q$ as $\varepsilon \to 0$. We let $u \in \mathcal{H}_T$ solve (1.16) with Cauchy data u_0, u_1, and set $u_\varepsilon = J_\varepsilon u$. Then, for any $\varepsilon > 0$, $u_\varepsilon \in C^0([0,T],H^4) \cap C^1([0,T],H^2) \cap C^2([0,T],L^2)$, and we also have that $u_\varepsilon \in L^q([0,T],L^r)$ when $n \geq 5$, where (q,r) is the B-admissible pair given by $q = 2(2^\sharp - 1)$ and $r = 2^\sharp(n+4)/(n+2)$. Moreover, u_ε solves the equation

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} + \Delta^2 u_\varepsilon = J_\varepsilon h(u) \tag{1.36}$$

with Cauchy data $J_\varepsilon u_0$ and $J_\varepsilon u_1$. We let $u_{\varepsilon,t}$ be the time derivative of u_ε. As is easily checked, $t \to E_0'\left(u_\varepsilon(t), u_{\varepsilon,t}(t)\right)$ is C^1, and by (1.36) we can write that

$$E_0'\left(u_\varepsilon(t_2), u_{\varepsilon,t}(t_2)\right) - E_0'\left(u_\varepsilon(t_1), u_{\varepsilon,t}(t_1)\right) = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} J_\varepsilon h(u) J_\varepsilon u_t dxdt \tag{1.37}$$

for all $t_1, t_2 \in (0,T)$. Since $J_\varepsilon v \to v$ in H^2 for $v \in H^2$, we also get that for any t,

$$E_0'\left(u_\varepsilon(t), u_{\varepsilon,t}(t)\right) \to E_0'\left(u(t), u_t(t)\right) \tag{1.38}$$

as $\varepsilon \to 0$. Following arguments in Cazenave [9] we can write that for any $1 < a, b < +\infty$, and any $v \in L^a([0,T], L^b)$,

$$J_\varepsilon v \to v \ \text{ in } L^a([0,T], L^b) \tag{1.39}$$

as $\varepsilon \to 0$. Similarly, we also get that for any $b > 1$, and any $v \in C^0([0,T], L^b)$,

$$J_\varepsilon v \to v \ \text{ in } C^0([0,T], L^b) \tag{1.40}$$

as $\varepsilon \to 0$. By the Strichartz's estimates of the linear theory in Lemma 1.1, see the remark after Lemma 1.1, and since $(2, 2^\star)$ for $2^\star = 2n/(n-2)$ is a S-admissible pair when $n \geq 3$, we have that $u_t \in L^2([0,T], L^{2^\star})$ when $n \geq 3$. Let $h_1 = \eta h$, and $h_2 = (1-\eta)h$ be as in (1.17). We clearly have that $h_1(u) \in C^0([0,T], L^2)$, that $h_2(u) \in C^0([0,T], L^2)$ when $n \leq 4$, and that $h_2(u) \in L^q([0,T], L^s)$ when $n \geq 5$, where s is such that $(2^\sharp - 1)s = r$ for q and r as above. By (1.39) and (1.40) we can write that $J_\varepsilon h_1(u) \to h_1(u)$ in $C^0([0,T], L^2)$ as $\varepsilon \to 0$, $J_\varepsilon h_2(u) \to h_2(u)$ in $C^0([0,T], L^2)$ as $\varepsilon \to 0$ when $n \leq 4$, and that $J_\varepsilon h_2(u) \to h_2(u)$ in $L^q([0,T], L^s)$ as $\varepsilon \to 0$ when $n \geq 5$. We also have that $J_\varepsilon u_t \to u_t$ in $C^0([0,T], L^2)$ as $\varepsilon \to 0$, and that $J_\varepsilon u_t \to u_t$ in $L^2([0,T], L^{2^\star})$ as $\varepsilon \to 0$ when $n \geq 3$. It follows that for any $t_1, t_2 \in (0,T)$,

$$\begin{aligned} &\int_{t_1}^{t_2}\int_{\mathbb{R}^n} J_\varepsilon h_1(u) J_\varepsilon u_t dxdt \to \int_{t_1}^{t_2}\int_{\mathbb{R}^n} h_1(u) u_t dxdt \text{ , and} \\ &\int_{t_1}^{t_2}\int_{\mathbb{R}^n} J_\varepsilon h_2(u) J_\varepsilon u_t dxdt \to \int_{t_1}^{t_2}\int_{\mathbb{R}^n} h_2(u) u_t dxdt \end{aligned} \tag{1.41}$$

as $\varepsilon \to 0$. When $n \geq 5$, and since $q \geq 2$, we can write that $h_2(u) \in L^2([0,T], L^s)$. By noting that $s = \frac{2n}{n+2}$ is the conjugate exponent of $2^\star$, and that $u_t \in L^2([0,T], L^{2^\star})$, it follows that $h_2(u)u_t \in L^1([0,T], L^1)$. By smoothing u with respect to the t-variable, since $u \in C^0([0,T], H^2) \cap C^1([0,T], L^2)$ and $u \in L^q([0,T], L^r)$, we can also prove that

$$\int_{\mathbb{R}^n} H_i\left(u(t_2)\right) dx - \int_{\mathbb{R}^n} H_i\left(u(t_1)\right) dx = \int_{t_1}^{t_2}\int_{\mathbb{R}^n} h_i(u) u_t dxdt \tag{1.42}$$

for all $t_1, t_2 \in (0,T)$, and for $i = 1, 2$, where, in this equation, $H_i(x) = \int_0^x h_i(t)dt$ for $x \in \mathbb{R}$. Combining (1.37), (1.38), (1.41), and (1.42), we get that

$$E'\left(u(t_2), u_t(t_2)\right) = E'\left(u(t_1), u_t(t_1)\right) \tag{1.43}$$

for all $t_1, t_2 \in (0,T)$. Since $t \to E'\left(u(t), u_t(t)\right)$ is continuous, it follows from (1.43) that $E'(u, u_t) = E'(u_0, u_1)$ in $[0,T]$. As already mentioned, this ends the proof of the conservation of the energy in Theorem 1.1. □

At this stage, in order to end the proof of Theorem 1.1, it remains to prove unconditional uniqueness of the solution, namely uniqueness in $\tilde{\mathcal{H}}_T$-spaces and not only in $\tilde{\mathcal{H}}_T \cap L^{q_n}([0,T], L^{r_n})$-spaces. The argument we use below was developed by Cazenave [9] for the Schrödinger equation. Related arguments can be found in Colliander, Keel, Staffilani, Takaoka, and Tao [13], Furioli and Terraneo [16], Furioli, Planchon, and Terraneo [17], and Kato [27]. We refer also to Tao and Visan [64].

Proof of Theorem 1.1 – Unconditional Uniqueness. Let f satisfy (1.1), $u_0 \in H^2$, and $u_1 \in L^2$. We prove that if u, v are two solutions of (0.1) in $\tilde{\mathcal{H}}_T$ for some $T > 0$, with Cauchy data u_0 and u_1, where $\tilde{\mathcal{H}}_T$ is as in (1.18), then $u = v$ in $[0, T]$. We may here assume that $n \geq 5$ since we already know by point (i) in the existence part of the proof of Theorem 1.1 that the result holds true when $n \leq 4$. In what follows we say that a pair (a, b) is a beam's intermediate pair if $a \geq 2$ and

$$\frac{2}{a} + \frac{n}{b} = \frac{n-2}{2} . \tag{1.44}$$

Let $T \in (0, 1]$, $k \in C^0([0, T], H^{-2}) \cap L^{a'}([0, T], L^{b'})$ for some beam's intermediate pair (a, b), and $u \in C^0([0, T], H^2) \cap C^1([0, T], L^2) \cap C^2([0, T], H^{-2})$ such that it solves the linear equation (1.7) in $C^0([0, T], H^{-2})$ with Cauchy data $u(0) = 0$ and $u_t(0) = 0$. Then

$$\|u\|_{L^\infty([0,T],H^1)} + \|u\|_{L^2([0,T],L^{2^\sharp})} \leq C\|k\|_{L^{a'}([0,T],L^{b'})} \tag{1.45}$$

for some positive constant $C > 0$ which does not depend on k and T. Such intermediate Strichartz type estimates follow from the Schrödinger structure of (0.1) and Strichartz estimates in Besov spaces for the Schrödinger equation. We may use, for instance, that

$$\begin{aligned} &\|v\|_{L^\infty([0,T],B^{-1}_{2,2})} \leq C\|k\|_{L^{a'}([0,T],B^{-1}_{c',2})} \text{ , and} \\ &\|v\|_{L^2([0,T],B^{-1}_{2^\star,2})} \leq C\|k\|_{L^{a'}([0,T],B^{-1}_{c',2})} , \end{aligned} \tag{1.46}$$

where $C > 0$ does not depend on k and T, c is such that the pair (a, c) is S-admissible, and the $B^s_{q_1,q_2}$ spaces are the standard Besov spaces. A possible reference for such estimates is Cazenave [9]. In particular, it follows from (1.45) that

$$\|u\|_{L^2([0,T],L^{2^\sharp})} + \|u\|_{L^\infty([0,T],L^2)} \leq C\|k\|_{L^{a'}([0,T],L^{b'})} , \tag{1.47}$$

where $C > 0$ does not depend on k and T. We let h, h_1, and h_2 be as in (1.17), and we consider (0.1) when written under the form of equation (1.16). We let also $T > 0$ and $u, v \in \mathcal{H}_T$ be two solutions of (1.16) satisfying the same Cauchy data u_0 and u_1, where $\mathcal{H}_T$ is as in (1.18). We set $w = v - u$. Then $w \in \mathcal{H}_T$ and w solves the equation (3.9) with Cauchy data $w(0) = 0$ and $w_t(0) = 0$, where $k = h(v) - h(u)$. For $M > 0$, we let $E = E_M$ be the subset of $[0, T] \times \mathbb{R}^n$ defined by $E_M = \{|u| + |v| \leq M\}$. We set $k_1 = h_1(v) - h_1(u) + \chi_E\, (h_2(v) - h_2(u))$, and $k_2 = \chi_{E^c}\, (h_2(v) - h_2(u))$, where χ_E and χ_{E^c} are the characteristic functions of E and E^c. As is easily checked, $k = k_1 + k_2$, and for $t \in (0, T)$, we can write by Hölder's inequality and (1.17) that $k_1 \in L^1([0, t], L^2)$, $k_2 \in L^2([0, t], L^{2n/(n+4)})$, and

$$\begin{aligned} &\|k_1\|_{L^1([0,t],L^2)} \leq C_1(1 + M^{p-2})t\|w\|_{L^\infty([0,t],L^2)} , \\ &\|k_2\|_{L^2([0,t],L^{2n/(n+4)})} \leq C_2\|\Psi\|^{p-2}_{L^\infty([0,T],L^{2^\sharp})}\|w\|_{L^2([0,t],L^{2^\sharp})} , \end{aligned} \tag{1.48}$$

where $\Psi = \chi_{E^c}\, (|u| + |v|)$, and $C_1, C_2 > 0$ do not depend on M and t. The pair $(2, \frac{2n}{n+4})$ is the conjugate pair of $(2, 2^\sharp)$ which satisfies the intermediate condition (1.44). We assume in what follows that $t \leq 1$. For $i = 1, 2$ we let w^i be the solution of the linear equation (1.7) with $k = k_i$ and Cauchy data $w^i(0) = 0$, $w^i_t(0) = 0$. We have $w^2 = w - w^1$ and it follows from Lemma 1.1 that the w^i's are all in $\mathcal{H}_T$. By the Strichartz estimates in Lemma 1.1 we can write that

$$\|w^1\|_{L^\infty([0,t],H^2)} \leq C_M t\|w\|_{L^\infty([0,t],L^2)} , \tag{1.49}$$

and by the intermediate Strichartz estimates (1.47) we can write that

$$\|w^2\|_{L^\infty([0,t],L^2)} + \|w^2\|_{L^2([0,t],L^{2^\sharp})} \le C\|\Psi\|^{p-2}_{L^\infty([0,T],L^{2^\sharp})}\|w\|_{L^2([0,t],L^{2^\sharp})} , \quad (1.50)$$

where $C_M > 0$ does not depend on t, and $C > 0$ does not depend on M and t. We let $M \gg 1$ be sufficiently large such that $C\|\Psi\|^{p-2}_{L^\infty([0,T],L^{2^\sharp})} < 1$. By combining (1.49) and (1.50), we then get that

$$\|w\|_{L^\infty([0,t],L^2)} \le C_M t\|w\|_{L^\infty([0,t],L^2)} , \quad (1.51)$$

where $C_M > 0$ does not depend on t. In particular, by choosing $t > 0$ sufficiently small, we get that $w = 0$ in $[0, t]$. Iterating the argument it follows that $w = 0$ in $[0, T]$ and this proves unconditional uniqueness. □

Theorem 1.1 easily follows from standard semi-group arguments, as developed in Cazenave and Haraux [10], when $p < 2^\sharp/2$ in (1.1). When f is assumed to be of class C^1, $|f'(u)|$ is dominated by $|u|^{p-1}$, and $p < 2^\sharp - 1$ is assumed to be subcritical, Theorem 1.1 was established by Levandosky [36]. The approach in [36] is based on the system representation of (0.1) and does not make use of the Schrödinger structure of the equation. Theorem 1.1 only needs (1.1) and, in particular, allows p to be critical.

2. Related results and remarks

We prove in this section various results related to the local existence theorem, Theorem 1.1 of Section 1. A first result which, together with Theorem 1.1 establishes well-posedness, is as follows.

Proposition 2.1. *Let f satisfy (1.1), $u_0 \in H^2$, $u_1 \in L^2$, $(u_k^0)_k$ be a sequence in H^2 converging to u_0 in H^2 as $k \to +\infty$, and $(u_k^1)_k$ be a sequence in L^2 converging to u_1 in L^2 as $k \to +\infty$. Let $T^\star = T^\star(u_0, u_1)$ be the maximal time of existence of the solution of (0.1) with Cauchy data u_0 and u_1, and $T_k^\star = T^\star(u_k^0, u_k^1)$ be the maximal time of existence of the solution of (0.1) with Cauchy data u_k^0 and u_k^1. Then,*

$$T^\star \le \liminf_{k\to+\infty} T_k^\star \quad (2.1)$$

and we also have that for any $T < T^\star$, $u_k \to u$ in $C^0([0,T],H^2) \cap C^1([0,T],L^2)$ as $k \to +\infty$, where u_k is the solution of (0.1) with Cauchy data u_k^0 and u_k^1, and u is the solution of (0.1) with Cauchy data u_0 and u_1.

Proof. We assume here that $n \ge 5$. The proof works the same, with only slight changes, when $n \le 4$. With the notations of the proposition, we let $\mathcal{U}^t$ be the solution of the homogeneous linear equation (1.27) with Cauchy data $\mathcal{U}^t(t) = u(t)$, $\mathcal{U}^t_t(t) = u_t(t)$. We let also $T < T^\star$ be fixed arbitrary. By the linear theory in Lemma 1.1 of Section 1 we can write that for any $\delta > 0$, there exists $\nu > 0$ such that

$$\|\mathcal{U}^t\|_{L^q([t,t+\nu],L^r)} < \delta/2 \quad (2.2)$$

for all $t \in [0, T]$, where $q = q_n$ and $r = r_n$ are as in (1.19). For $\tilde u_0, \tilde u_0' \in H^2$, and $\tilde u_1, \tilde u_1' \in L^2$, we denote by $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{U}}'$ the solutions of the homegeneous linear equation (1.27) with Cauchy data $\tilde u_0$, $\tilde u_1$, and $\tilde u_0'$, $\tilde u_1'$. Also we denote by $\tilde u$ and $\tilde u'$ the solutions of the nonlinear equation (0.1) with Cauchy data $\tilde u_0$, $\tilde u_1$, and $\tilde u_0'$, $\tilde u_1'$. We fix $\varepsilon_0 > 0$. By coming back to point (ii) in the proof of local existence in Section 1, and by the linear theory in Lemma 1.1, we get that there exists $\delta > 0$ small such that if $\|\tilde{u}_0' - \tilde{u}_0\|_{H^2} + \|\tilde{u}_1' - \tilde{u}_1\|_{L^2} \le \varepsilon_0$, $\|\tilde{\mathcal{U}}\|_{L^q([0,\tilde{T}],L^r)} < \delta$, and $\|\tilde{\mathcal{U}}'\|_{L^q([0,\tilde{T}],L^r)} < \delta$ for some $\tilde{T} \in (0,1)$, then $\tilde{u}, \tilde{u}' \in \mathcal{H}$, $\tilde{u}$ and $\tilde{u}'$ solve (0.1) in $C^0([0,T_0],H^{-2})$, and

$$\|\tilde{u}' - \tilde{u}\|_{\mathcal{H}} < C\left(\|\tilde{u}_0' - \tilde{u}_0\|_{H^2} + \|\tilde{u}_1' - \tilde{u}_1\|_{L^2}\right) \tag{2.3}$$

where $C > 0$ depends only on n, $\mathcal{H} = \mathcal{H}_{T_0} \cap L^q([0,T_0],L^r)$, $\mathcal{H}_{T_0}$ is as in (1.18), and $T_0 = \delta\tilde{T}/\sqrt{1 + E_0(\tilde{u}_0,\tilde{u}_1)}$. We fix such a $\delta > 0$ and let $\Lambda_0 = \max_{[0,T]} E_0(u,u_t)$ and $\tilde{\nu} = \delta\nu/2\sqrt{1+\Lambda_0}$, where ν is given by (2.2). We let also $t_1 < t_2 < \cdots < t_N$ be such that $t_1 = 0$, $t_N = T$, $|t_{i+1} - t_i| < \tilde{\nu}$ for all $i = 1,\dots,N$, and $[0,T] = \bigcup_{i=1}^{N-1}[t_i,t_{i+1}]$. By combining (2.2) and (2.3), thanks also to the linear theory in Lemma 1.1 of Section 1, we get that if the u_k's of the proposition exist on $[t_i, t_i+\varepsilon]$ for some $\varepsilon > 0$ and some $i = 1,\dots,N$, and if it holds that $u_k(t_i) \to u(t_i)$ in H^2 and $u_{k,t}(t_i) \to u_t(t_i)$ in L^2 as $k \to +\infty$, where $u_{k,t}$ stands for the time derivative of u_k, then u_k exists in $[t_i, t_i + \tilde{\nu}]$ for $k \gg 1$, and

$$\|u_k - u\|_{C^0([t_i,t_i+\tilde{\nu}],H^2)} + \|u_k - u\|_{C^1([t_i,t_i+\tilde{\nu}],L^2)} \to 0 \tag{2.4}$$

as $k \to +\infty$. Since $t_1 = 0$ we know by assumption that $u_k(t_1) \to u(t_1)$ in H^2 and that $u_{k,t}(t_1) \to u_t(t_1)$ in L^2 as $k \to +\infty$. By (2.4) we can iterate from t_1 to t_{N-1}. This clearly ends the proof of the proposition. □

Proposition 2.2 below is concerned with the explosion of the energy for low dimensions and in the energy-subcritical case $p < 2^\sharp - 1$ when $n \ge 5$. For the sake of simplicity, an additional condition on f we require in the second part of the proposition is that there exist $\mu \in (0,m)$ and $C > 0$ such that

$$F(x) \le \frac{\mu}{2}x^2 + C|x|^{p+1} \tag{2.5}$$

for all $x \in \mathbb{R}$. Various f satisfy (1.1) and (2.5). Any f satisfying (1.1) and (3.1), as in Sections 3 to 5, satisfy (1.1) and (2.5). Proposition 2.2 states as follows. As a remark, we do have that $\frac{n}{4}(p-1) < p+1$ as soon as $p < 2^\sharp - 1$.

Proposition 2.2. *Let f satisfy (1.1) with $p < 2^\sharp - 1$ when $n \ge 5$. Let also $T^\star = T^\star(u_0,u_1)$ be the maximal time of existence of the solution u of (0.1) with Cauchy data u_0 and u_1. If $T^\star < +\infty$, then $\|u\|_{H^2} \to +\infty$ as $t \to T^\star$, and if we assume (2.5), then it also holds that $\|u\|_{L^q} \to +\infty$ as $t \to T^\star$ for all $q \in [2,+\infty)$ such that $q > p-1$ when $1 \le n \le 4$, and all $q \in [2,2^\sharp]$ such that $q > \frac{n}{4}(p-1)$ when $n \ge 5$.*

Proof. Let f satisfy (1.1) . We know from Theorem 1.1 that the total energy E is conserved, and from the remark following the proof of existence in Section 1 that if $T^\star < +\infty$, then $E_0(u,u_t) \to +\infty$ as $t \to T^\star$. Taking $y = 0$ in (1.1) we have that $|F(x)| \le C(|x|^2 + |x|^{p+1})$ for all x. In particular, by the embedding $H^2 \subset H^{s,2}$ for $s \le 2$, and the Sobolev embedding theorem for fractional spaces, we get that $\|u\|_{H^2} \to +\infty$ as $t \to T^\star$ if $T^\star < +\infty$. Now we assume that f also satisfies (2.5). By (2.5), Hölder's inequality, conservation of the total energy, and the Sobolev embedding theorem, we can write that

$$\begin{aligned}\|u\|_{H^2}^2 &\le 2E(u_0,u_1) + 2\int_{\mathbb{R}^n} F(u)dx \\ &\le \mu\|u\|_2^2 + C\left(1 + \|u\|_{L^{p+1}}^{p+1}\right) \\ &\le \frac{\mu}{m}\|u\|_{H^2}^2 + C\left(1 + \|u\|_{L^q}^{(p+1)\theta}\|u\|_{L^{q'}}^{(p+1)(1-\theta)}\right) \\ &\le \frac{\mu}{m}\|u\|_{H^2}^2 + C\left(1 + \|u\|_{L^q}^{(p+1)\theta}\|u\|_{H^2}^{(p+1)(1-\theta)}\right)\end{aligned} \tag{2.6}$$

for all $t \ge 0$, where $1 \le q \le p+1 \le q'$, q' can be chosen arbitrarily large if $n \le 4$, $q' = 2n/(n-4)$ if $n \ge 5$,

$$\theta = \frac{\frac{1}{p+1} - \frac{1}{q'}}{\frac{1}{q} - \frac{1}{q'}}\ , \tag{2.7}$$

and $C > 0$ does not depend on t. As is easily checked from (2.7), by choosing $q' \gg 1$ sufficiently large we get that $(1-\theta)(p+1) < 2$ if $q > p-1$ and $1 \le n \le 4$. In a similar way, when $n \ge 5$, we get that $(1-\theta)(p+1) < 2$ if $q > \frac{n}{4}(p-1)$. Coming back to (2.6), since $\mu < m$ and $\|u\|_{H_2^2} \to +\infty$ as $t \to T^\star$, we necessarily have that $\|u\|_{L^q} \to +\infty$ as $t \to T^\star$ if $(1-\theta)(p+1) < 2$. This proves that $\|u\|_{L^q} \to +\infty$ as $t \to T^\star$ for all $q \in [2, p+1]$ such that $q > p-1$ when $1 \le n \le 4$, and $q > \frac{n}{4}(p-1)$ when $n \ge 5$. By Hölder's inequality, coming back to (2.6), we can also write that

$$\begin{aligned}\|u\|_{H^2}^2 &\le 2E(u_0,u_1) + 2\int_{\mathbb{R}^n} F(u)dx \\ &\le \mu\|u\|_2^2 + C\left(1 + \|u\|_{L^{p+1}}^{p+1}\right) \\ &\le \frac{\mu}{m}\|u\|_{H^2}^2 + C\left(1 + \|u\|_{L^2}^{(p+1)\theta}\|u\|_{L^q}^{(p+1)(1-\theta)}\right) \\ &\le \frac{\mu}{m}\|u\|_{H^2}^2 + C'\left(1 + \|u\|_{H^2}^{(p+1)\theta}\|u\|_{L^q}^{(p+1)(1-\theta)}\right)\end{aligned} \tag{2.8}$$

for all $t \ge 0$, where $2 \le p+1 \le q$ and $(p+1)\theta = 2(q-p-1)/(q-2)$. As is easily checked, $(p+1)\theta < 2$, and it follows from (2.8) that $\|u\|_{L^q} \to +\infty$ as $t \to T^\star$ for all $q \ge p+1$. This ends the proof of the proposition. □

A corollary to Proposition 2.2 is as follows. The condition $F(x) \le Cx^2$ in Corollary 2.1 is automatically satisfied if f is Lipschitz, or if there exists $C > 0$ such that $xf(x) \le C'x^2$ for all x. As a remark, the arguments in Segal [52], see also Shatah and Struwe [55], can be transposed with basically no changes to (0.1). In particular, see Section 7, (0.1) possesses a weak solution of finite energy defined in the whole of $\mathbb{R}$, with Cauchy data $u_0 \in H^2$ and $u_1 \in L^2$, as soon as $f(0) = 0$, f is locally Lipschitz, $xf(x) \le 0$ for all x, and $F(u_0) \in L^1$. Corollary 2.1 states as follows.

Corollary 2.1. *Let f satisfy (1.1) with $p < 2^\sharp - 1$ when $n \ge 5$, and let F be the primitive of f as in (1.3). Assume there exists $C > 0$ such that $F(x) \le Cx^2$ for all x. Then, for any $u_0 \in H^2$ and $u_1 \in L^2$, the solution u of (0.1) with Cauchy data u_0 and u_1 exists for all $t \in \mathbb{R}$.*

Proof. By reversing time it suffices to prove existence for all $t \ge 0$. We fix $u_0 \in H^2$, $u_1 \in L^2$, and let u be the solution of (0.1) with Cauchy data u_0, u_1. By (3.7) in

Section 3,

$$\begin{aligned}\|u(t)\|_{L^2}^2 &= \|u(0)\|_{L^2}^2 + 2\int_0^t \int_{\mathbb{R}^n} u(s)u_t(s)dsdx \\ &\le \|u(0)\|_{L^2}^2 + 2\frac{m+1}{m}\int_0^t E_0\left(u(s),u_t(s)\right)ds\end{aligned} \tag{2.9}$$

for all $t \geq 0$. By the conservation of the total energy in Theorem 1.1, and since by assumption $F(x) \leq Cx^2$, we then get with (2.9) that

$$\begin{aligned}E_0\left(u(t),u_t(t)\right) &= E_0(u_0,u_1) - \int_{\mathbb{R}^n} F(u_0)dx + \int_{\mathbb{R}^n} F\left(u(t)\right)dx \\ &\le E_0(u_0,u_1) - \int_{\mathbb{R}^n} F(u_0)dx + C\int_{\mathbb{R}^n} u(t)^2 dx \\ &\le C_1\int_0^t E_0\left(u(s),u_t(s)\right)ds + C_2\end{aligned} \tag{2.10}$$

for all $t \geq 0$, where $C_1, C_2 > 0$ are positive constants which do not depend on t. In particular, by the integral form of Gronwall's inequality, we get with (2.10) that

$$E_0\left(u(t),u_t(t)\right) \le C_2\left(1 + C_1 t e^{C_1 t}\right) \tag{2.11}$$

for all $t \geq 0$, and by (2.11) we get that $E_0\left(u(t),u_t(t)\right)$ remains bounded on any time interval $[0,T]$. By Theorem 1.1, this implies that u exists on the whole half line $\mathbb{R}^+$, namely for all $t \geq 0$. This proves the corollary. □

The model case for the Breteherton equation (0.1) is given by the pure power nonlinearity $f(x) = \lambda|x|^{p-1}x$, $p > 1$. In that case the equation writes as

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + mu = \lambda|u|^{p-1}u\ , \tag{2.12}$$

where $\lambda \in \mathbb{R}\backslash\{0\}$. The equation is defocusing when $\lambda < 0$, and focusing when $\lambda > 0$. A straightforward consequence of Corollary 2.1 and of Proposition 2.1 is that the defocusing nonlinear equation (2.12) is globally well-posed in $C^0(\mathbb{R},H^2)\cap C^1(\mathbb{R},L^2)$ for all $p > 1$ when $n \leq 4$, and all $p \in (1, 2^\sharp - 1)$ when $n \geq 5$. The model equation (2.12) has scaling invariance

$$\begin{aligned}&u(t,x) \to \frac{1}{\lambda^{\frac{4}{p-1}}}u\left(\frac{t}{\lambda^2},\frac{x}{\lambda}\right)\ ,\ u_0(x) \to \frac{1}{\lambda^{\frac{4}{p-1}}}u_0\left(\frac{x}{\lambda}\right)\ , \\ &u_1(x) \to \frac{1}{\lambda^{\frac{2(p+1)}{p-1}}}u_1\left(\frac{x}{\lambda}\right)\ ,\ \text{and } m \to \frac{m}{\lambda^4}\ .\end{aligned} \tag{2.13}$$

In the energy-critical case, where $n \geq 5$ and $p = 2^\sharp - 1$, the scaling preserves energy. The following proposition shows that for general energy-critical equations, where $n \geq 5$ and $p = 2^\sharp - 1$ in (1.1), thus including the focusing case in (2.12), blow-up of mass holds in mixed norms.

Proposition 2.3. *Let $n \geq 5$ and f satisfy (1.1) with $p = 2^\sharp - 1$. Let $u_0 \in H^2$, $u_1 \in L^2$, and $T^\star = T^\star(u_0,u_1)$ be the maximal time of existence of the solution of (0.1) with Cauchy data u_0 and u_1. If $T^\star < +\infty$, then*

$$\lim_{T\to T^\star}\int_0^T \|u\|_{L^{r_n}}^{q_n} dt = +\infty\ , \tag{2.14}$$

where q_n and r_n are given by $q_n = 2(2^\sharp - 1)$ and $r_n = 2^\sharp(n+4)/(n+2)$.

Proof. We prove (2.14) by contradiction. We assume in what follows that $T^\star < +\infty$ and that the limit in the left hand side of (2.14) is finite. A preliminary claim in that case is that we also have that

$$\sup_{[0,T^\star)} E_0(u,u_t) < +\infty \ , \tag{2.15}$$

where E_0 is as in (1.3). In order to prove (2.15) we let $\varepsilon > 0$ sufficiently small to be chosen later on, and let $I_j = [t_j, t_{j+1}]$, $j = 1,\dots,N$, be a family of closed intervals such that $t_1 = 0$, $t_j < t_{j+1} < t_j + \varepsilon$ for all j, and $t_{N+1} \ge T^\star$. We let also h, $h_1 = \eta h$, and $h_2 = (1-\eta)h$ be as in (1.17), and we consider (0.1) when written under the form of equation (1.16). We can write that $u = U_j + V_{1,j} + V_{2,j}$ in $\tilde I_j$, where U_j is the solution of (1.27) with Cauchy data $u(t_j)$ and $u_t(t_j)$ at $t = t_j$, the $V_{i,j}$'s are the solutions of (1.7) with $k = h_i(u)$ and Cauchy data 0 and 0 at $t = t_j$ for $i = 1, 2$, and $\tilde I_j = I_j$ when $j < N$ while $\tilde I_N = [t_N, T^\star)$. By the linear theory in Lemma 1.1 of Section 1, and by inequlaities like in (1.26), we can write that for any j,

$$\begin{aligned} &\|u\|_{C^0(\tilde I_j,H^2)} + \|u_t\|_{C^0(\tilde I_j,L^2)} \\ &\le C\left(\sqrt{E_0\left(u(t_j),u_t(t_j)\right)} + \varepsilon\|u\|_{C^0(\tilde I_j,H^2)} + K^{(2^\sharp-1)/q_n}\right) \ , \end{aligned} \tag{2.16}$$

where $C > 0$ depends only on n, f, and m, and K is the left hand side in (2.14). By assumption, $K < +\infty$, and by choosing $\varepsilon > 0$ sufficiently small such that $C\varepsilon < 1$, it follows from (2.16) that (2.15) holds true. Now we let $T < T^\star$ sufficiently close to $T^\star$ to be chosen later on, and let $\mathcal{U}_T$ be the solution of (1.27) with Cauchy data $u(T)$ and $u_t(T)$ at $t = T$, the $V_{i,T}$'s are the solutions of (1.7) with $k = h_i(u)$ and Cauchy data 0 and 0 at $t = T$ for $i = 1, 2$. By writing that $\mathcal{U}_T = u - V_{1,T} - V_{2,T}$, by the linear theory in Lemma 1.1 of Section 1, and by (1.26), it holds that

$$\begin{aligned} &\|\mathcal{U}_T\|_{L^{q_n}([T,T^\star],L^{r_n})} \\ &\le C\left(\|u\|_{L^{q_n}([T,T^\star),L^{r_n})} + (T^\star - T)K' + \|u\|^{2^\sharp-1}_{L^{q_n}([T,T^\star),L^{r_n})}\right) \ , \end{aligned} \tag{2.17}$$

where $C > 0$ depends only on n, f, and m, and $K' = \sup_{[0,T^\star)} E_0(u,u_t)$ is finite by (2.15). By (2.17) we then get that

$$\lim_{T\to T^\star} \|\mathcal{U}_T\|_{L^{q_n}([T,T^\star],L^{r_n})} = 0 \tag{2.18}$$

and it follows from (2.18) that for any $\delta > 0$, there exist $T < T^\star$ and $\varepsilon > 0$ such that $\|\mathcal{U}\|_{L^{q_n}([T,T^\star+\varepsilon],L^{r_n})} < \delta$. By (1.34) and (2.15) we then get that u can be extended on an interval like $[T, T^\star + \varepsilon']$ for some $T < T^\star$ and $\varepsilon' > 0$. A contradiction with the definition of $T^\star$. This proves (2.14) and Proposition 2.3. □

A natural question on Theorem 1.1 concerns the existence of a lower bound for the maximal time $T^\star$ of existence of a solution of (0.1) with Cauchy data u_0, u_1. As is to be expected, the time of existence for critical equations should depend on the profile of the initial data and not simply on the energy. In the subcritical case the situation is easier to handle. We already know, see (1.33) after the proof of existence in Section 1, that if $p < 2^\sharp - 1$ when $n \ge 5$, and $p \ge (2^\sharp - 1)n/(n+2)$ which we can always assume without loss of generality, then there exists $K = K(f,n,m) \in (0,1)$, depending only on f, n, and m, such that for any $u_0 \in H^2$ and $u_1 \in L^2$,

$$T^\star \geq \frac{K}{\left(1+\sqrt{E_0(u_0,u_1)}\right)^{\frac{p-1}{\delta}}} , \tag{2.19}$$

where $\delta = 1$ if $n \leq 4$ and $\delta > 0$ is as in (1.21) if $n \geq 5$. We point out here that more information than in (2.19) can be obtained if we assume that $p \leq \frac{n}{n-4}$ in (1.1) when $n \geq 5$. More precisely, we claim that if f satisfies (1.1), , where $p > 1$ is arbitrary when $n \leq 4$ and $p \leq \frac{n}{n-4}$ when $n \geq 5$, and if f is k_0 times differentiable at 0 and such that $f^{(k)}(0) = 0$ for all $0 \leq k \leq k_0$ and some $k_0 \leq p-1$, then there exist $C > 0$ such that

$$T^\star \geq \frac{C}{\varepsilon^{k_0/2}} \text{ if } k_0 > 0 \text{ and } T^\star \geq C|\log\varepsilon| \text{ if } k_0 = 0 \tag{2.20}$$

for all $\varepsilon \in \left(0, \frac{1}{2}\right)$ and all Cauchy data $u_0 \in H^2$ and $u_1 \in L^2$ such that $E_0(u_0,u_1) < \varepsilon$, where $T^\star$ is the maximal time of existence of the solution of (0.1) with Cauchy data u_0, u_1. We prove (2.20) as follows. By the conservation of the energy in Theorem 1.1, $\frac{d}{dt}E_0\left(u(t),u_t(t)\right) = \int_{\mathbb{R}^n} f\left(u(t)\right)u_t(t)dx$, and it follows from Hölder's inequality that if $N(t) = E_0\left(u(t),u_t(t)\right)^{1/2}$, then

$$N(t) \leq N(0) + \int_0^t \|f\left(u(s)\right)\|_{L^2}ds \tag{2.21}$$

for $t \geq 0$. By the assumption on f that $f^{(k)}(0) = 0$ for all $0 \leq k \leq k_0$ and some $k_0 \leq p-1$, and by (1.1), we can write that $|f(x)| \leq C\left(|x|^{k_0+1} + |x|^p\right)$ for all $x \in \mathbb{R}$, where $C > 0$ is independent of x. By the Sobolev embedding theorem we then get that $\|f\left(u(t)\right)\|_{L^2} \leq C\left(N(t)^{k_0+1} + N(t)^p\right)$, where $C > 0$ is independent of t. Assuming that $N(0) < 1$, by continuity of N, we do get that $N(t) \leq 1$ for $t > 0$ small. Let $t_0 > 0$ be the upper bound of the set consisting of the positive t wich are such that $N(s) \leq 1$ for all $0 \leq s \leq t$. By (2.21), and according to the above remarks, we can write that

$$N(t) \leq N(0) + 2C\int_0^t N(s)^{k_0+1}ds \tag{2.22}$$

for all $0 \leq t \leq t_0$, where $C > 0$ does not depend on t. Let Φ be the function of t in the right hand side of (2.22). By (2.22), $\Phi'(t) \leq 2C\Phi(t)^{k_0+1}$ and $N(t) \leq \Phi(t)$ for all $0 \leq t \leq t_0$. Assume $k_0 = 0$. Then $\Phi'(t) \leq 2C\Phi(t)$, and we get that

$$\Phi(t) \leq \Phi(0)e^{2Ct} \leq \sqrt{\varepsilon}e^{2Ct} \tag{2.23}$$

since $\phi(0) = N(0)$ and $E_0(u_0,u_1) < \varepsilon$. In particular, $\Phi(t) < 1$, and hence $N(t) < 1$, if $4Ct < |\ln\varepsilon|$. By the definition of t_0, this implies that $4Ct_0 \geq |\ln\varepsilon|$, and since $t_0 \leq T$, (2.20) holds true when $k_0 = 0$. When $k_0 \geq 1$, since $\Phi'(t) \leq 2C\Phi(t)^{k_0+1}$, we get in a smiliar way that $2k_0Ct_0 \geq C'\varepsilon^{-k_0/2}$ when $E_0(u_0,u_1) < \varepsilon$ and $2\varepsilon < 1$, where $C' > 0$ depends only on k_0. This proves (2.20) when $k_0 \geq 1$. As a remark, the bound $p \leq \frac{n}{n-4}$ is the energy bound which makes that $f(u) \in L^1([0,T],L^2)$ and that the material in Cazenave and Haraux [10] can be applied. On the other hand, no conditions on p are required if $n \leq 4$.

Global well-posedness for the energy-critical defocusing wave equation was established few years ago. The case of radially-symmetric initial data goes back to Struwe [58]. The case of arbitrary initial data is due to Grillakis [19, 20], and

Shatah and Struwe [53, 54]. Related references are Struwe [59], and Shatah and Struwe [55]. Global well-posedness for the energy-critical Schrödinger equation was established only very recently. The case of radially-symmetric initial data is due to Bourgain [5] in dimension $n = 3$, see also Grillakis [21], and to Tao [62] in arbitrary dimensions. The case of arbitrary initial data is due to Colliander, Keel, Staffilani, Takaoka, and Tao [13] in dimension $n = 3$, Ryckman and Visan [50] in dimension $n = 4$, and Visan [65] when the dimension $n \geq 5$. A recent very interesting survey on the subject is Tao [63].

3. Blow-up in finite time

We let f be such that (1.1) holds true. We also assume that there exists $\varepsilon > 0$ such that

$$xf(x) \geq (2+\varepsilon)F(x) \tag{3.1}$$

for all $x \in \mathbb{R}$, where F is the primitive of f as in (1.3). Various f satisfy (1.1) and (3.1). The nonlinearity of the focusing model equation (2.12), given by $f(x) = \lambda|x|^{p-1}x$ where $\lambda > 0$, satisfies (1.1) and (3.1) when p is as in (1.1). We aim here in proving blow-up in finite time for solutions of (0.1). As a preliminary result, following Cazenave [8], we claim that when f satisfies (1.1) and (3.1), then $F^+(x) = O(|x|^{2+\varepsilon})$ as $x \to 0$, and we can write that for any $\mu < m$, there exists $C > 0$ such that

$$F(x) \leq \frac{\mu}{2}x^2 + C|x|^{p+1} \tag{3.2}$$

for all $x \in \mathbb{R}$, where $F^+ = \max(0, F)$. We prove (3.2) as follows. We let h be the function defined for $x \neq 0$ by $h(x) = |x|^{-(2+\varepsilon)}F(x)$. Then,

$$h'(x) = \frac{x}{|x|^{4+\varepsilon}}\left(xf(x) - (2+\varepsilon)F(x)\right) \tag{3.3}$$

for all $x \neq 0$, and it follows from (3.1) that h is non increasing in $(-\infty, 0)$ and non decreasing in $(0, +\infty)$. As an easy consequence we get that $F^+(x) \leq C_1|x|^{2+\varepsilon}$ for all x such that $|x| \leq 1$, where $C_1 = \max(F^+(-1), F^+(1))$. In particular, the first part of the above claim holds true. Integrating (1.1), we also get that there exists $C > 0$ such that $|F(x)| \leq C\left(|x|^2 + |x|^{p+1}\right)$ for all x. It follows that $|F(x)| \leq C_2|x|^{p+1}$ for all x such that $|x| \geq 1$, where $C_2 > 0$ does not depend on x, and we can write that

$$F(x) \leq C\left(|x|^{2+\varepsilon} + |x|^{p+1}\right) \tag{3.4}$$

for all $x \in \mathbb{R}$, where $C > 0$ is given by $C = \max(C_1, C_2)$. As is easily checked, (3.2) follows from (3.4) and the property that $|F(x)| = O(|x|^{p+1})$ as $|x| \to +\infty$. This proves the above claim that for any $\mu < m$, there exists $C > 0$ such that (3.2) holds true. Now, for u a solution of (0.1) with Cauchy data u_0, u_1, we let $L_2 = L_{u_0,u_1}$ be the square L^2-norm function defined by

$$L_2(t) = \int_{\mathbb{R}^n} u^2(t)dx \tag{3.5}$$

and we let also $H = H_{u_0,u_1}$ be the function given by

$$H(t) = L_2(t) - \frac{2(2+\varepsilon)}{\varepsilon m}E(u_0, u_1)\ . \tag{3.6}$$

An easy claim is that L_2, and hence also H, are C^2-functions. We have here that

$$\begin{aligned} &L_2'(t) = 2\int_{\mathbb{R}^n} u(t)u_t(t)dx \text{ , and} \\ &L_2''(t) = 2\langle u_{tt}(t), u(t)\rangle_{H^{-2}\times H^2} + 2\int_{\mathbb{R}^n} u_t^2(t)dx \text{ ,} \end{aligned} \tag{3.7}$$

where $\langle\cdot,\cdot\rangle_{H^{-2}\times H^2}$ is the pairing between H^{-2} and H^2. Moreover, by (3.6), $H'(t) = L_2'(t)$ and $H''(t) = L_2''(t)$ for all t. When $H(t) \geq 0$ and $H'(t) > 0$ for some $t \geq 0$, or when $H(t) > 0$ and $H'(t) \geq 0$, we define $T^\sharp(t) = T^\sharp_{u_0,u_1}(t)$ by

$$\begin{aligned} &T^\sharp(t) = t + \frac{4}{\varepsilon}\sqrt{\frac{(4+\varepsilon)L_2(t)}{\varepsilon m H(t)}} - \frac{(4+\varepsilon)L_2'(t)}{\varepsilon^2 m H(t)} \text{ if } S_H(t) < 0 \text{ , and} \\ &T^\sharp(t) = t + \frac{4L_2(t)}{\varepsilon L_2'(t)} \text{ if } S_H(t) \geq 0 \text{ ,} \end{aligned} \tag{3.8}$$

where $S_H(t) = \frac{L_2'(t)^2}{L_2(t)H(t)} - \frac{4\varepsilon m}{4+\varepsilon}$, $L_2 = L_{u_0,u_1}$ is as in (3.5), and $H = H_{u_0,u_1}$ is as in (3.6). By the conservation of the total energy in Theorem 1.1, and since $E(u,v) \geq 0$ if $u \equiv 0$, we get that $E(u_0,u_1) \geq 0$ if $L_2(t) = 0$ for some $t \geq 0$. We also have that $L_2'(t) = 0$ if $L_2(t) = 0$. In particular, $L_2(t) > 0$ if $H(t) \geq 0$ and $H'(t) > 0$, or $H(t) > 0$ and $H'(t) \geq 0$. By convention, we let $S_H(t) = +\infty$ if $H(t) = 0$. A solution u of (0.1) is said to blow up in finite time if its maximal time of existence $T^\star$, also referred to as its lifespan, is finite.

Lemma 3.1. *Let f satisfy (1.1) and (3.1), and let u be a solution of (0.1) with Cauchy data u_0, u_1. Let $H = H_{u_0,u_1}$ be as in (3.6). Suppose there exists $t_0 \geq 0$ such that*

$$\begin{aligned} &H(t_0) \geq 0 \textit{ and } H'(t_0) > 0 \text{ , } \textit{or} \\ &H(t_0) > 0 \textit{ and } H'(t_0) \geq 0 \text{ .} \end{aligned} \tag{3.9}$$

Then u blows up in finite time with a lifespan $T^\star \leq T^\sharp(t_0)$, where $T^\sharp = T^\sharp_{u_0,u_1}$ is as in (3.8).

Proof. By combining (0.1), (3.1), and the conservation of the total energy in Theorem 1.1, we get with (3.7) that

$$\begin{aligned} H''(t) &\geq \varepsilon\int_{\mathbb{R}^n}\left((\Delta u)^2 + mu^2\right)dx + (4+\varepsilon)\int_{\mathbb{R}^n} u_t^2 dx - 2(2+\varepsilon)E(u_0,u_1) \\ &\geq \varepsilon m H(t) + (4+\varepsilon)\int_{\mathbb{R}^n} u_t^2 dx \geq \varepsilon m H(t) \end{aligned} \tag{3.10}$$

for all t. Let $t_0 \geq 0$ be such that (3.9) holds true. By (3.10), since $H''(t) \geq \varepsilon m H(t)$ for all t, we can write that $H(t) > 0$ for all $t > t_0$. We also have that $H'(t) > 0$ for all $t > t_0$. Once again by (3.10), we then get that $H''(t) \geq (4+\varepsilon)\|u_t\|^2_{L^2}$ for all $t \geq t_0$, and we get with (3.7) that

$$\begin{aligned} L_2'(t)^2 &\leq 4\|u\|^2_{L^2}\|u_t\|^2_{L^2} \\ &\leq \frac{4}{4+\varepsilon}L_2(t)L_2''(t) \end{aligned} \tag{3.11}$$

for all $t \geq t_0$ since $L_2'' = H''$. Let us assume from now on that t_0 is such that $H(t_0) \geq 0$ and $H'(t_0) > 0$. We have that $L_2(t) > 0$ for all $t \geq t_0$. Let K be the function given by $K(t) = L_2(t)^{-\varepsilon/4}$ for $t \geq t_0$. Then

$$K''(t) = \frac{\varepsilon}{4L_2(t)^{2+\frac{\varepsilon}{4}}} \left(\frac{\varepsilon+4}{4} L_2'(t)^2 - L_2(t) L_2''(t) \right) \tag{3.12}$$

and, by (3.11), we get that $K''(t) \leq 0$ for all $t \geq t_0$. In particular, we can write that $K(t) \leq K(t_0) + K'(t_0)(t - t_0)$ for all $t \geq t_0$, and since we also have that $K(t) \geq 0$, we get that

$$t \leq t_0 + \frac{4L_2(t_0)}{\varepsilon L_2'(t_0)} . \tag{3.13}$$

This proves that if $t_0 \geq 0$ is such that $H(t_0) \geq 0$ and $H'(t_0) > 0$, then u blows up in finite time with a lifespan $T^\star$ bounded from above by the right hand side in (3.13). We assume from now on that $t_0 \geq 0$ is such that $H(t_0) > 0$, $H'(t_0) \geq 0$, and $S_H(t_0) < 0$. As already mentionned, we can write that $H(t_0 + s) > 0$ and $H'(t_0 + s) > 0$ for $s > 0$. In particular, it follows from what we just proved that u blows up in finite time with a lifespan $T^\star$ bounded from above by

$$T^\star \leq t_0 + s + \frac{4L_2(t_0 + s)}{\varepsilon L_2'(t_0 + s)} \tag{3.14}$$

for $s > 0$. Since $L_2''(t) \geq \varepsilon m H(t)$ for $t \geq t_0$, $H(t) \geq 0$ and $H'(t) \geq 0$ for $t \geq t_0$, and $L_2'(t) = H'(t)$ for all t, we have that

$$\begin{aligned} &L_2(t_0 + s) \leq L_2(t_0) + sL_2'(t_0 + s) \text{ , and} \\ &L_2'(t_0 + s) \geq L_2'(t_0) + s \min_{t \in [t_0, t_0+s]} H''(t) \geq L_2'(t_0) + s\varepsilon m H(t_0) . \end{aligned} \tag{3.15}$$

By combining (3.14) and (3.15), it follows that

$$\begin{aligned} &T^\star \leq t_0 + T_{t_0}(s) \text{ , where} \\ &T_{t_0}(s) = \left(1 + \frac{4}{\varepsilon}\right) s + \frac{4L_2(t_0)}{\varepsilon \left(L_2'(t_0) + s\varepsilon m H(t_0)\right)} . \end{aligned} \tag{3.16}$$

Let s_0 be given by

$$s_0 = \sqrt{\frac{4L_2(t_0)}{\varepsilon(\varepsilon+4)mH(t_0)}} - \frac{L_2'(t_0)}{\varepsilon m H(t_0)} . \tag{3.17}$$

Then $s_0 > 0$ if $S_H(t_0) < 0$. The function T_{t_0} in (3.16) is decreasing up to s_0, and increasing after s_0. By (3.16) we can write that if $t_0 \geq 0$ is such that $H(t_0) > 0$, $H'(t_0) \geq 0$, and $S_H(t_0) < 0$, then u blows up in finite time with a lifespan $T^\star$ bounded from above by $T^\star \leq t_0 + T_{t_0}(s_0)$, where s_0 is given by (3.17). Noting that $t_0 + T_{t_0}(s_0)$ is precisely the right hand side of the first equation in (3.8) when $t = t_0$, and that $t_0 + T_{t_0}(s_0)$ is less than the right hand side in (3.13), this ends the proof of the lemma. □

Several situations where blow-up occurs can be obtained with Lemma 3.1. In particular, Theorem 3.1 below holds true. Theorem 3.1 for Klein-Gordon equations in domains of the Euclidean space was proved in Cazenave [8]. The negative energy part in Theorem 3.1 was proved in Levine [38] in such a general setting that it includes the present situation. Possible related references on such kind of results are John [25] and Strauss [57].

Theorem 3.1. *Let f satisfy (1.1) and (3.1), and let u be a solution of (0.1) with Cauchy data u_0, u_1. Suppose that one of the three following conditions (i)-(iii) is satisfied :*

(i) $E(u_0,u_1) < 0$, or $E(u_0,u_1) = 0$ and $u \not\equiv 0$,

(ii) $(u_0,u_1)_{L^2\times L^2} > \frac{2+\varepsilon}{\sqrt{(4+\varepsilon)\varepsilon m}} E(u_0,u_1)$,

(iii) $(u_0,u_1)_{L^2\times L^2} + \Lambda(m)H(0) > 0$,

where E is the total energy as in (1.3), $H = H_{u_0,u_1}$ is as in (3.6), $\Lambda(m) = \sqrt{\frac{\varepsilon m}{8}}$ if $H(0) \geq 0$, and $\Lambda(m) = \sqrt{\frac{\varepsilon m}{2}}$ if $H(0) \leq 0$. Then u blows up in finite time.

Proof. Suppose first that $E(u_0,u_1) < 0$. By contradiction we assume that u exists for all $t \geq 0$. Since $E(u_0,u_1) < 0$, we have that $H(t) > 0$ for all $t \geq 0$. By (3.10), we also have that $H''(t) \geq 2(2+\varepsilon)|E(u_0,u_1)|$ for all $t \geq 0$. It clearly follows from such an inequality that $H'(t) > 0$ for $t \gg 1$ large, a contradiction by Lemma 3.1. This proves that u blows up in finite time if $E(u_0,u_1) < 0$. Suppose now that $E(u_0,u_1) = 0$ and that $(u_0,u_1) \not\equiv (0,0)$. Clearly $H(t) \geq 0$ for all $t \geq 0$. By contradiction we assume that u exists for all $t \geq 0$. Then, by Lemma 3.1, $H'(t) \leq 0$ for all $t \geq 0$. By (3.10), $H''(t) \geq 0$ for all $t \geq 0$, and H' is nondecreasing. Since $H(t) \geq 0$ for all $t \geq 0$, we get that $H'(t) \to 0$ as $t \to +\infty$. We have that $L_2'(t) = H'(t)$ and $L_2''(t) = H''(t)$ for all $t \geq 0$ (and even that $L_2 = H$ in the present context). With (3.10) we can write that $L_2''(t) \geq 2\varepsilon E_0(u,u_t)$ for all $t \geq 0$, where E_0 is as in (1.3). In particular,

$$\int_{t_1}^{t_2} E_0(u,u_t)dt \leq \frac{1}{2\varepsilon}|L_2'(t_1)| \tag{3.18}$$

for all $t_1 < t_2$, and we get that there exists a sequence $(t_k)_k$ such that $t_k \to +\infty$ as $k \to +\infty$, and such that $E_0\left(u(t_k),u_t(t_k)\right) \to 0$ as $k \to +\infty$. By the conservation of the total energy in Theorem 1.1 we also have that $E\left(u(t_k),u_t(t_k)\right) = 0$ for all k. Now we can use (3.2) with $\mu = m/2$, and the Sobolev embedding theorem, to write that there exists $C > 0$ such that for all k,

$$\begin{aligned} 0 &= E\left(u(t_k),u_t(t_k)\right) \\ &\geq \frac{1}{2}E_0\left(u(t_k),u_t(t_k)\right) - C\|u(t_k)\|_{L^{p+1}}^{p+1} \\ &\geq \left(\frac{1}{2}+o(1)\right)E_0\left(u(t_k),u_t(t_k)\right) , \end{aligned} \tag{3.19}$$

where $o(1) \to 0$ as $k \to +\infty$. For $k \gg 1$ sufficiently large, (3.19) is impossible. This proves that u blows up in finite time if $E(u_0,u_1) = 0$ and $(u_0,u_1) \not\equiv (0,0)$. In particular, point (i) in Theorem 3.1 is proved. Now we assume that u_0 and u_1 are such that the strict inequality in point (ii) of Theorem 3.1 is satisfied. Then $\Phi(0) > 0$, where $\Phi(t)$ for $t \geq 0$ is given by

$$\Phi(t) = \frac{1}{2}L_2'(t) - \frac{2+\varepsilon}{\sqrt{\varepsilon m(4+\varepsilon)}}E(u_0,u_1) . \tag{3.20}$$

By contradiction we assume that u exists for all $t \geq 0$. From (3.7) we easily deduce that

$$|L_2'(t)| \leq \frac{\varepsilon m}{\sqrt{\varepsilon m(4+\varepsilon)}}L_2(t) + \frac{\sqrt{\varepsilon m(4+\varepsilon)}}{\varepsilon m}\|u_t\|_{L_2}^2 \tag{3.21}$$

for all $t \ge 0$. By combining (3.10) and (3.21) we then get that

$$L_2''(t) \ge \sqrt{\varepsilon m(4+\varepsilon)}\,|L_2'(t)| - 2(2+\varepsilon)E(u_0,u_1) \tag{3.22}$$

for all $t \ge 0$. Then, by (3.22), $\Phi'(t) \ge \sqrt{\varepsilon m(4+\varepsilon)}\Phi(t)$ for all $t \ge 0$, where $\Phi(t)$ is given by (3.20). It follows from Gronwall's inequality that

$$\Phi(t) \ge \Phi(0)e^{\sqrt{\varepsilon m(4+\varepsilon)}t} \tag{3.23}$$

for all $t \ge 0$. Since we assumed that $\Phi(0) > 0$, we get with (3.23) that $L_2'(t) \to +\infty$ as $t \to +\infty$. In particular, $L_2(t) \gg 1$ and $L_2'(t) > 0$ for $t \gg 1$ large, and we get a contradiction with Lemma 3.1. Point (ii) in Theorem 3.1 is proved. It remains to prove (iii). We let u_0 and u_1 be as in point (iii) of Theorem 3.1. By contradiction we assume that u exists for all $t \ge 0$, and we distinguish the two cases $H(0) \ge 0$ and $H(0) < 0$, where $H = H_{u_0,u_1}$ is given by (3.6). First we assume that $H(0) \ge 0$. Then, by (iii) and Lemma 3.1, $H'(0) < 0$ and $H(0) > 0$. Since $H''(t) \ge \varepsilon m H(t)$, H' is nondecreasing in any time interval $[0,t_1)$ where H remains nonnegative, and we can write that $H(t) \ge H(0) + tH'(0)$ in $[0,t_1)$. It follows that H remains nonnegative at least up to the time $t_0 = H(0)/|H'(0)|$. By (iii), $t_0 > \sqrt{2/(\varepsilon m)}$ while, since $H''(t) \ge \varepsilon m H(t)$, we get that

$$\begin{aligned} H'(t) &\ge H'(0) + \varepsilon m \int_0^t (H(0) + H'(0)s)\,ds \\ &= H'(0)\left(1 + \frac{\varepsilon m}{2}t^2\right) + \varepsilon m H(0)t \end{aligned} \tag{3.24}$$

for all $t \in [0,t_0]$. In particular, $H'(t_0) > 0$, and since we also have that $H(t_0) \ge 0$, the contradiction follows from Lemma 3.1. This proves (iii) when $H(0) \ge 0$ and we may now assume that $H(0) < 0$. Then $H'(0) > 0$. Since $H''(t) \ge \varepsilon m H(t)$, and H is nondecreasing when $H' \ge 0$, we can write that $H'(t) \ge H'(0) + t\varepsilon m H(0)$ in any time interval $[0,t_1)$ where H' remains nonnegative. It follows that H' remains nonnegative at least up to the time $t_0 = H'(0)/(\varepsilon m|H(0)|)$. By (iii), $t_0 > \sqrt{2/(\varepsilon m)}$ while

$$\begin{aligned} H(t) &\ge H(0) + \int_0^t (H'(0) + \varepsilon m H(0)s)\,ds \\ &= H(0) + tH'(0) + \frac{\varepsilon m H(0)}{2}t^2 \end{aligned} \tag{3.25}$$

for all $t \in [0,t_0]$. In particular, $H(t_0) > 0$, and since we also have that $H'(t_0) \ge 0$, the contradiction follows from Lemma 3.1. Point (iii) in Theorem 3.1 is proved. □

By Lemma 3.1 we also get explicit upper bounds for the lifespan of u in Theorem 3.1. For instance, if $E(u_0,u_1) < 0$, then the lifespan $T^\star$ of u is such that

$$\begin{aligned} T^\star &\le \frac{(u_0,u_1)^-_{L^2\times L^2}}{(2+\varepsilon)|E(u_0,u_1)|} + \frac{4}{\varepsilon}\sqrt{\frac{(4+\varepsilon)\|u_0\|^2_{L^2}}{\varepsilon m\|u_0\|^2_{L^2} - 2(2+\varepsilon)E(u_0,u_1)}} \\ &\le \frac{(u_0,u_1)^-_{L^2\times L^2}}{(2+\varepsilon)|E(u_0,u_1)|} + \frac{4}{\varepsilon}\sqrt{\frac{4+\varepsilon}{\varepsilon m}}\,, \end{aligned} \tag{3.26}$$

where E is as in (1.3), $(u_0,u_1)_{L^2\times L^2}$ is the L^2-scalar product of u_0 with u_1, and $\|u_0\|_{L^2}$ is the L^2-norm of u_0. We prove (3.26) as follows. Since $E(u_0,u_1) < 0$, we have that $H(t) > 0$ for all t, where $H = H_{u_0,u_1}$ is given by (3.6). If $H'(0) \ge 0$, we immediately get (3.26) with Lemma 3.1. If not the case, $H'(0) < 0$. By

(3.10) we have that $H''(t) \geq 2(2+\varepsilon)|E(u_0,u_1)|$ for all t. Then we can write that $H'(t) \geq H'(0) + 2(2+\varepsilon)|E(u_0,u_1)|t$, and if $t_0 > 0$ is such that $H'(t) < 0$ for all $t < t_0$ and $H'(t_0) = 0$, we get that

$$t_0 \leq \frac{(u_0,u_1)^-_{L^2\times L^2}}{(2+\varepsilon)|E(u_0,u_1)|} . \tag{3.27}$$

Then $H(t_0) > 0$, $H'(t_0) \geq 0$, and by Lemma 3.1, u blows up in finite time with a lifespan $T^\star$ bounded from above by $T^\sharp(t_0)$, where $T^\sharp$ is given by (3.8). Since $H'(t) \leq 0$ for $t \leq t_0$, we have that $L_2(t_0) \leq L_2(0)$, and (3.26) follows from the bound $T^\star \leq T^\sharp(t_0)$ and the bound (3.27) on t_0. Similar upper bounds can be obtained in the other cases of Theorem 3.1.

4. Small initial data

We aim here in proving that if u is a solution of (0.1) with Cauchy data u_0, u_1, and if $E_0(u_0,u_1)$ is small, where E_0 is as in (1.3), then u exists for all t and the kinetic energy $E_0\left(u(t),u_t(t)\right)$ at all time is controlled by the kinetic energy $E_0(u_0,u_1)$ at time $t=0$. Theorem 4.1 for Klein-Gordon equations in domains of the Euclidean space was proved in Cazenave [8]. The result was recently emphasized in Keel and Tao [29].

Theorem 4.1. *Let f satisfy (1.1) and (3.1). Then there exists $\delta > 0$ and a function $K \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ with $K(0)=0$ such that for any $(u_0,u_1) \in H^2\times L^2$ of kinetic energy $E_0(u_0,u_1) < \delta$, the solution u of (0.1) with Cauchy data u_0,u_1 exists for all $t\in\mathbb{R}$, and satisfies that $E_0\left(u(t),u_t(t)\right) \leq K\left(E_0(u_0,u_1)\right)$ for all t.*

Proof. By (1.3) and (3.2) with $\mu = \frac{m}{2}$, there exists $C > 0$ such that for any $u \in H^2$ and any $v \in L^2$,

$$\begin{aligned} E(u,v) &\geq E_0(u,v) - \frac{m}{4}\int_{\mathbb{R}^n} u^2 dx - C\int_{\mathbb{R}^n} |u|^{p+1}dx \\ &\geq \frac{1}{2}E_0(u,v) - C\int_{\mathbb{R}^n}|u|^{p+1}dx . \end{aligned}$$

By Sobolev embeddings it follows that there exists $C_1 > 0$ such that for any $u \in H^2$ and any $v \in L^2$,

$$E(u,v) \geq \frac{1}{2}E_0(u,v) - C_1E_0(u,v)^{\frac{p+1}{2}} . \tag{4.1}$$

Letting $y=0$ in (1.1) we get that $|f(x)| \leq C\left(|x| + |x|^p\right)$ for all $x \in \mathbb{R}$, where $C > 0$ is independent of x. By Sobolev embeddings and (1.3), integrating this inequality, we also get that there exists $C_2 > 1$ such that for any $u \in H^2$ and any $v \in L^2$,

$$E(u,v) \leq C_2E_0(u,v)\left(1 + E_0(u,v)^{\frac{p-1}{2}}\right) . \tag{4.2}$$

Let $h_1,h_2 : \mathbb{R}^+ \to \mathbb{R}$ be the functions of $E_0(u,v)$ we have in the right hand sides of (4.1) and (4.2). Namely, $h_1(t) = \frac{1}{2}t - C_1t^{(p+1)/2}$ and $h_2(t) = C_2t(1+t^{(p-1)/2})$. Both h_1 and h_2 are C^1-functions on $\mathbb{R}^+$. We have that $h_1(0) = h_2(0) = 0$, $h_1'(0) > 0$, and $h_2'(0) > 0$. We let $\delta_1 > 0$ be such that $h_1'(t) > 0$ for all $0 \leq t \leq 2\delta_1$, and let $\delta_2 \in (0,\delta_1)$ be such that $h_2'(t) > 0$ for all $0 \leq t \leq \delta_2$, and $h_2(\delta_2) < h_1(\delta_1)$. We let $h \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ be such that $h = h_1^{-1}$ in $[0, h_1(2\delta_1)]$. Let $u_0 \in H^2$ and $u_1 \in L^2$ be such that $E_0(u_0,u_1) < \delta_2$. By (4.2) we get that $E(u_0,u_1) < h_2(\delta_2)$, and we can write that $E(u_0,u_1) < h_1(\delta_1)$. We also have that $E_0(u_0,u_1) < \delta_1$. Let u be the solution of (0.1) with Cauchy data u_0, u_1. The function $t \to E_0(u, u_t)$ is continuous. For $t \geq 0$ small, $E_0(u, u_t) < \delta_1$. We claim here that for any $t \in [0, T^\star)$, $E_0(u, u_t) < \delta_1$, where $T^\star$ is the maximal time of existence of u. Indeed if $E_0(u, u_t) \geq \delta_1$ at $t = t_0$ for some $t_0 > 0$, then there exists some possibly other time $t_1 > 0$ such that $E_0(u, u_t) \in [\delta_1, 2\delta_1)$ at $t = t_1$. By (4.1), since h_1 is increasing on $[0, 2\delta_1]$, it follows that $h_1(\delta_1) \leq h_1 \left(E_0(u, u_t)\right) \leq E(u, u_t)$ at $t = t_1$. By the conservation of the total energy in Theorem 1.1, $E(u, u_t) = E(u_0, u_1)$. Since $E(u_0, u_1) < h_1(\delta_1)$, the contradiction follows. This proves the above claim that if $E_0(u_0, u_1) < \delta_2$, then $E_0(u, u_t) < \delta_1$ for all $t \in [0, T^\star)$. When $n \leq 4$, or $p < 2^\sharp - 1$ and $n \geq 5$, we then get by Proposition 2.2 that $T^\star = +\infty$, and thus that u exists for all $t \geq 0$. In the critical case, where $n \geq 5$ and $p = 2^\sharp - 1$, we let $\delta > 0$ be as in (1.34). Let also $\delta_1 > 0$ be such that $C\sqrt{\delta_1} \leq \delta$ for some $C > 0$, $T < T^\star$ be such that

$$T + \frac{\delta}{\sqrt{1 + \delta_1}} > T^\star ,$$

and $\mathcal{U}$ be the solution of the linear equation (1.27) with Cauchy data $u(T)$ and $u_t(T)$. By the Strichartz estimates in Lemma 1.1 of Section 1, there exists $C_0 > 0$, independent of T, such that

$$\|\mathcal{U}\|_{L^q(I,L^r)} \leq C_0 \sqrt{E_0\left(u(T), u_t(T)\right)} \tag{4.3}$$

for all interval $I \subset \mathbb{R}^+$ of length $|I| \leq 1$ such that $T \in I$, where $q = q_n$ and $r = r_n$ are as in (1.19). Letting $C = C_0$ it easily follows from (1.34) that, here again, in the critical case, we must have that $T^\star = +\infty$ and that u exists for all $t \geq 0$. Now, by the conservation of the total energy, and by (4.2),

$$E(u, u_t) = E(u_0, u_1) \leq h_2\left(E_0(u_0, u_1)\right) \tag{4.4}$$

for all $t \geq 0$. We also have that $h_2\left(E_0(u_0, u_1)\right) < h_2(\delta_2)$ and $h_2(\delta_2) < h_1(2\delta_1)$ when $E_0(u_0, u_1) < \delta_2$. In particular, by (4.1) and (4.4), $E_0(u, u_t) \leq K\left(E_0(u_0, u_1)\right)$ for all $t \geq 0$, where $K = h \circ h_2$ is C^1 on $[0, \delta_2]$ and can thus be regarded as being defined and C^1 on the whole of $\mathbb{R}^+$. It remains to prove similar estimates for $t \leq 0$. For $u_0 \in H^2$ and $u_1 \in L^2$, we let v be the solution of (0.1) with Cauchy data u_0, $-u_1$. If $E_0(u_0, u_1) < \delta_2$, we get with the above discussion that v exists for all $t \geq 0$ and that $E_0(v, v_t) \leq K\left(E_0(u_0, u_1)\right)$ for all $t \geq 0$. Let w be defined by $w(t, \cdot) = u(t, \cdot)$ if $t \geq 0$, and $w(t, \cdot) = v(-t, \cdot)$ if $t \leq 0$. Clearly, if we still denote by w the map $t \to w(t, \cdot)$, then $w \in C^0(\mathbb{R}, H^2)$, $w \in C^1(\mathbb{R}, L^2)$, $w_{|t=0} = u_0$, and $w_{t|t=0} = u_1$. As is easily checked, since u and v solves (0.1) in $\mathbb{R}^+$, we also get that $w \in C^2(\mathbb{R}, H^{-2})$ and that w solves (0.1) in $\mathbb{R}$. This proves the Theorem. □

In the spirit of Theorem 4.1, using slightly different arguments, a complete picture can be given in the particular case of the model equation (2.12) when p is subcritical. We assume in what follows that $p < 2^\sharp - 1$ when $n \geq 5$ and consider (2.12) for which $f(x) = \lambda |x|^{p-1} x$. We let $u_0 \in H^2$, $u_1 \in L^2$, and u be the solution of (2.12) with Cauchy data u_0 and u_1. We already know by Corollary 2.1 that in the defocusing case, where $\lambda < 0$, u exists for all $t \in \mathbb{R}$. By Sobolev embeddings, and conservation of the energy, we also get that $E_0(u, u_t) \leq K\left(E_0(u_0, u_1)\right)$ for all t, where $K(s) = s + \Lambda s^{(p+1)/2}$ for some $\Lambda > 0$. In the focusing case, where $\lambda > 0$, we let

$$\begin{aligned} \mathcal{S} &= \left\{(u, v) \in H^2 \times L^2 \text{ s.t. } E(u, v) \leq \delta_0 \text{ and } I(u) \geq 0\right\} , \\ \mathcal{S}' &= \left\{(u, v) \in H^2 \times L^2 \text{ s.t. } E(u, v) < \delta_0 \text{ and } I(u) < 0\right\} , \end{aligned} \tag{4.5}$$

where E is as in (1.3), $I(u) = \|u\|_{H^2}^2 - \lambda\|u\|_{L^{p+1}}^{p+1}$, $\|\cdot\|_{H^2}$ is as in (1.2), δ_0 is given by

$$\delta_0 = \frac{(p-1)K_p^{\frac{p+1}{p-1}}}{2(p+1)\lambda^{\frac{2}{p-1}}}\ , \tag{4.6}$$

and K_p is the sharp constant for subcritical embeddings defined as the infimum over u in $H^2\backslash\{0\}$ of the ratio $\|u\|_{H^2}^2/\|u\|_{L^{p+1}}^2$. By Hölder's inequality we have that $\|u\|_{L^{p+1}} \le \|u\|_{L^{2^\sharp}}^\theta \|u\|_{L^2}^{1-\theta}$, where $\theta \in (0,1)$ is given by $4(p+1)\theta = n(p-1)$. In particular, we can write that $K_p \ge m^{1-\theta}K_n^\theta$, where K_n is the infimum over u in $H^2\backslash\{0\}$ of the ratio $\|\Delta u\|_{L^2}^2/\|u\|_{L^{2^\sharp}}^2$. The exact value of K_n was computed in Beckner [2], Edmunds, Fortunato, and Janelli [14], Lieb [39], and Lions [42]. It is given by $K_n = n(n-4)(n^2-4)\pi^2\Gamma(n/2)^{4/n}\Gamma(n)^{-4/n}$, where Γ is the Euler function. Following Payne-Sattinger [48] and Sattinger [51] we easily get that the following proposition holds true.

Proposition 4.1. *Assume $p < 2^\sharp - 1$ when $n \ge 5$ and consider (2.12) with $\lambda > 0$. For $u_0 \in H^2$ and $u_1 \in L^2$, let u be the solution of (2.12) with Cauchy data u_0 and u_1. Given $\delta_0 > 0$ as in (4.6), the sets $\mathcal{S}$ and $\mathcal{S}'$ in (4.5) are stable. Moreover, if $(u_0,u_1) \in \mathcal{S}$, then u exists for all $t \in \mathbb{R}$, and*

$$\frac{1}{2}\int_{\mathbb{R}^n} u_t^2 dx + \frac{p-1}{2(p+1)}\int_{\mathbb{R}^n}\left((\Delta u)^2 + mu^2\right)dx \le E(u_0,u_1) \tag{4.7}$$

for all $t \in \mathbb{R}$, and if $(u_0,u_1) \in \mathcal{S}'$, then u blows up in finite time.

As is easily checked, for any $(u,v) \in H^2 \times L^2$, we do have here, in the model case, that $E(u,v) \le E_0(u,v)$ and that if $E_0(u,v) \le \delta_0^2$, then $I(u) \ge 0$. In particular, it follows from Proposition 4.1 that when $f(x) = \lambda|x|^{p-1}x$ for some $\lambda > 0$, and $p < 2^\sharp - 1$ if $n \ge 5$, then we can take $\delta = \min\left(\delta_0, \delta_0^2\right)$ in Theorem 4.1. Proposition 4.1 provides an explicit value for δ in Theorem 4.1.

Proof. First we prove that $\mathcal{S}$ is stable and that if $(u_0,u_1) \in \mathcal{S}$, then u exists for all $t \in \mathbb{R}$, and (4.7) holds true for all t. Let $u \in H^2$, $v \in L^2$, and I be as in (4.5). A first remark is that if $I(u) \ge 0$, then we can write that

$$\begin{aligned} E(u,v) &= \frac{1}{2}\int_{\mathbb{R}^n}\left((\Delta u)^2 + mu^2\right)dx - \frac{\lambda}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx + \frac{1}{2}\int_{\mathbb{R}^n} v^2 dx \\ &\ge \frac{p-1}{2(p+1)}\int_{\mathbb{R}^n}\left((\Delta u)^2 + mu^2\right)dx + \frac{1}{2}\int_{\mathbb{R}^n} v^2 dx\ . \end{aligned} \tag{4.8}$$

In particular, $E_0(u,v)$ is bounded from above by $\frac{p+1}{p-1}E(u,v)$, and the estimate (4.7) follows from (4.8) and the stability of $\mathcal{S}$. Another remark is that if $I(u) = 0$, then

$$\int_{\mathbb{R}^n}\left((\Delta u)^2 + mu^2\right)dx = \lambda\int_{\mathbb{R}^n}|u|^{p+1}dx \ge K_p\left(\int_{\mathbb{R}^n}|u|^{p+1}dx\right)^{\frac{2}{p+1}} \tag{4.9}$$

and we get a lower bound for the L^{p+1}-norm of u if $u \not\equiv 0$. In particular, by (4.8) and (4.9), we can write that

$$\begin{aligned} E(u,v) &\ge \frac{(p-1)\lambda}{2(p+1)}\int_{\mathbb{R}^n}|u|^{p+1}dx + \frac{1}{2}\int_{\mathbb{R}^n} v^2 dx \\ &\ge \frac{(p-1)\lambda}{2(p+1)}\int_{\mathbb{R}^n}|u|^{p+1}dx \ge \delta_0 \end{aligned} \tag{4.10}$$

for all $u \in H^2\backslash\{0\}$ and all $v \in L^2$ when $I(u) = 0$. Furthemore,

$$I(u) \geq \|u\|_{H^2}^2 \left(1 - \lambda \frac{\|u\|_{H^2}^{(p-1)/2}}{K_p^{(p+1)/2}}\right) \geq 0 \tag{4.11}$$

when $\|u\|_{H^2} \leq \delta_0$. Now we prove the stability of $\mathcal{S}$. Let $(u_0, u_1) \in \mathcal{S}$, and let u be the solution of (0.1) with Cauchy data u_0, u_1. First we assume that $E(u_0, u_1) < \delta_0$. The function $t \to I(u(t))$ is continuous and nonnegative at $t = 0$. By contradiction we assume that there exists $t_0 > 0$ such that $(u(t_0), u_t(t_0)) \notin \mathcal{S}$. By conservation of the energy, $E(u(t_0), u_t(t_0)) = E(u_0, u_1) < \delta_0$. It follows that $I(u(t_0)) < 0$. We let $t_1 \in [0, t_0)$ be such that $I(u(t_1)) = 0$ and $I(u(t)) < 0$ for $t \in (t_1, t_0)$. By (4.10) and conservation of the energy, we have that $u(t_1) = 0$, and since $I \geq 0$ in a neighbourhood of 0 by (4.11), we get a contradiction. In particular, if $(u_0, u_1) \in \mathcal{S}$ is such that $E(u_0, u_1) < \delta_0$, then $(u, u_t) \in \mathcal{S}$ for all t. By (4.8) we then get that $E_0(u, u_t)$ remains bounded, and it follows from Proposition 2.2 that u exists for all $t \geq 0$. Now we assume that $E(u_0, u_1) = \delta_0$. Suppose that for some time $t_0 \geq 0$, $I(u) = 0$ at time $t = t_0$. Then either $u(t_0) \equiv 0$, and in that case $I(u(t)) \geq 0$ for $t \geq t_0$ close to t_0 by (4.11), or $u(t_0) \not\equiv 0$. When $u(t_0) \not\equiv 0$, we get by (4.10) that $u_t(t_0) \equiv 0$ and that $u = u(t_0)$ is a minimizer for K_p. In particular, since $I(u) = 0$ at $t = t_0$, we get that $u = u(t_0)$ is a stationnary solution of (0.1). By the uniqueness of the solution in Theorem 1.1, it follows that $(u, u_t) = (u(t_0), 0)$ for all $t \geq t_0$. In that case, $I(u) = 0$ for all $t \geq t_0$, and it follows from the discussion that if $E(u_0, u_1) = \delta_0$ and if for some time $t_0 \geq 0$, $I(u) = 0$ at time $t = t_0$, then $I(u(t)) \geq 0$ for $t \geq t_0$ close to t_0. In particular, the set consisting of the $t \geq 0$ such that $(u(t), u_t(t)) \in \mathcal{S}$ is both open and closed in $[0, T^\star)$. As a consequence, $(u, u_t) \in \mathcal{S}$ for all t. By (4.8), as above, we then get that $E_0(u, u_t)$ remains bounded, and it follows from Proposition 2.2 that u exists for all $t \geq 0$. Summarizing, $\mathcal{S}$ is stable, and if $(u_0, u_1) \in \mathcal{S}$ then the solution u of (0.1) with Cauchy data u_0, u_1 exists for all $t \geq 0$. Moreover, by (4.8), we also have that (4.7) holds true for all $t \geq 0$. If $(u_0, u_1) \in \mathcal{S}$, then $(u_0, -u_1) \in \mathcal{S}$. By reversing time, u exists for all $t \in \mathbb{R}$. Now we prove that $\mathcal{S}'$ is also stable, but with the property that if $(u_0, u_1) \in \mathcal{S}'$, then the solution u of (0.1) with Cauchy data u_0 and u_1 blows up in finite time. The stability of $\mathcal{S}'$ easily follows from the conservation of the total energy in Theorem 1.1, (4.11), and the remark in (4.10) that $E(u, u_t) \geq \delta_0$ if $I(u) = 0$ and $u \not\equiv 0$. In particular, starting with $(u_0, u_1) \in \mathcal{S}'$, we get that $(u, u_t) \in \mathcal{S}'$ for all t, where u is the solution of (0.1) with Cauchy data u_0, u_1. It remains to prove that u blows up in finite time. Let L_2 be as in (3.5). By (3.7), since u solves (0.1),

$$L_2''(t) = 2\int_{\mathbb{R}^n} u_t^2 dx - 2I(u) \tag{4.12}$$

for all $t \geq 0$. In particular, since $(u, u_t) \in \mathcal{S}'$, we get that $L_2''(t) \geq 0$ for all $t \geq 0$. If $L_2'(t_1) > 0$ for some $t_1 \geq 0$, then $L_2'(t) \geq L_2'(t_1)$ for all $t \geq t_1$, and if u exists for $t \gg t_1$ sufficiently large with respect to t_1, we get that $H(t) > 0$ for $t \gg t_1$, where H is as in (3.6). In particular, u blows up in finite time by Lemma 3.1. We may therefore assume in what follows that $L_2'(t) \leq 0$ for all $t \geq 0$. In particular, L_2 is nonincreasing and nonnegative, while L_2' is nondecreasing and nonpositive. By contradiction we also assume that u exists for all $t \in \mathbb{R}^+$. Then, as is easily checked, $L_2(t) \to \alpha$ and $L_2'(t) \to 0$ as $t \to +\infty$, where $\alpha \geq 0$, and we also get that $L_2''(t_k) \to 0$ as $k \to +\infty$ for a sequence $(t_k)_k$ such that $t_k \to +\infty$ as $k \to +\infty$. By

(4.12) it follows that

$$\int_{\mathbb{R}^n} u_t(t_k)^2 dx \to 0 \ \text{ and } \ I\left(u(t_k)\right) \to 0 \tag{4.13}$$

as $k \to +\infty$. By the definition of K_p and (4.6) we can write that if $u \in H^2 \backslash \{0\}$ is such that $I(u) \le 0$, so in particular if $u \in H^2$ is such that $I(u) < 0$, then

$$\begin{aligned} \delta_0 &\le \frac{p-1}{2(p+1)} \left(\frac{\int_{\mathbb{R}^n} \left((\Delta u)^2 + mu^2\right) dx}{\left(\lambda \int_{\mathbb{R}^n} |u|^{p+1} dx\right)^{2/(p+1)}} \right)^{\frac{p+1}{p-1}} \\ &\le \frac{p-1}{2(p+1)} \left(\frac{\lambda \int_{\mathbb{R}^n} |u|^{p+1} dx + I(u)}{\left(\lambda \int_{\mathbb{R}^n} |u|^{p+1} dx\right)^{2/(p+1)}} \right)^{\frac{p+1}{p-1}} \\ &\le \frac{(p-1)\lambda}{2(p+1)} \int_{\mathbb{R}^n} |u|^{p+1} dx \,. \end{aligned} \tag{4.14}$$

Letting $u = u(t_k)$ in (4.14), it follows from (4.13) and (4.14) that

$$\begin{aligned} \delta_0 &\le E\left(u(t_k), u_t(t_k)\right) - \frac{1}{2} \int_{\mathbb{R}^n} u_t(t_k)^2 dx - \frac{1}{2} I\left(u(t_k)\right) \\ &\le E\left(u(t_k), u_t(t_k)\right) + o(1) \end{aligned} \tag{4.15}$$

for all k, where $o(1) \to 0$ as $k \to +\infty$. By the conservation of the total energy in Theorem 1.1 we would get with (4.15) that $E(u_0, u_1) \ge \delta_0$, a contradiction. In particular, u blows up in finite time, and this ends the proof of the proposition. □

In the critical case, where $p = 2^\sharp - 1$ when $n \ge 5$, it remains true that both $\mathcal{S}$ and $\mathcal{S}'$ are stable, that if $(u_0, u_1) \in \mathcal{S}'$, then u blows up in finite time, and that if $(u_0, u_1) \in \mathcal{S}$, then $E_0(u, u_t)$ remains bounded in $[0, T^\star)$. In particular, if $E_0(u_0, u_1) \le \delta_0$, then $E_0(u, u_t)$ remains bounded as long as u exists. Important advances in the radially symmetric case for the focusing energy-critical Schrödinger equation have been obtained in the recent Kenig and Merle [30]. Kenig and Merle [31] also recently solved the case of the focusing energy-critical wave equation in dimensions $3 \le n \le 5$.

5. Uniform bounds

We aim in this section in proving uniform energy bounds for solutions of (0.1) which exist on the half line $\mathbb{R}^+$, or the whole line $\mathbb{R}$. Such bounds have already been proved in Theorem 4.1 and Proposition 4.1 in the case of small energy data, with an explicit expression for the bound in Proposition 4.1. A consequence of the mathematics developed in Section 3 is that we also have an explicit expression for the bound when we restrict our attention to the L^2-norm of the solution. More precisely, the following proposition holds true.

Proposition 5.1. *Let f satisfy (1.1) and (3.1), and let u be a solution of (0.1) with Cauchy data u_0, u_1. If u exists on the half line $\mathbb{R}^+$, then*

$$\int_{\mathbb{R}^n} u^2 dx \le \frac{2(2+\varepsilon)}{\varepsilon m} E(u_0, u_1) + \min\left(H(0)^+, \frac{2\left(u_0, u_1\right)^-_{L^2 \times L^2}}{\varepsilon m t} \right) \tag{5.1}$$

for all $t \ge 0$, where H is as in (3.6), $(u_0, u_1)_{L^2 \times L^2}$ is the L^2-scalar product of u_0 with u_1, and E is the total energy as in (1.3).

Another way we can write (5.1) in Proposition 5.1 is that if f satisfies (1.1) and (3.1), and if u the solution of (0.1), with Cauchy data u_0 and u_1, exists on the half line $\mathbb{R}^+$, then

$$\begin{aligned} &\int_{\mathbb{R}^n} u^2 dx \le \max\left(\|u_0\|_{L^2}^2, \frac{2(2+\varepsilon)}{\varepsilon m} E(u_0,u_1)\right) \text{ , and} \\ &\int_{\mathbb{R}^n} u^2 dx \le \frac{2(2+\varepsilon)}{\varepsilon m} E(u_0,u_1) + \frac{2\,(u_0,u_1)^-_{L^2\times L^2}}{\varepsilon m t} \end{aligned} \tag{5.2}$$

for all $t \ge 0$, where E is as in (1.3), $(u_0,u_1)_{L^2\times L^2}$ is the L^2-scalar product of u_0 with u_1, and $\|u_0\|_{L^2}$ is the L^2-norm of u_0. The first equation in (5.2) is of interest when $t \ge 0$ is small. The second equation in (5.2) is of interest when $t \ge 0$ is large.

Proof. We prove (5.2). By Lemma 3.1, $H'(t) \le 0$ when $H(t) \ge 0$, and $H'(t) < 0$ when $H(t) > 0$. It follows that $H(t) \le H(0)^+$ for all $t \ge 0$. This proves that the first equation in (5.2) holds true. By Theorem 3.1 we clearly have that $E(u_0,u_1) \ge 0$ while, by Lemma 3.1, $H(0) \le 0$ if $(u_0,u_1)_{L^2\times L^2} \ge 0$. In particular, the second equation in (5.2) reduces to the first equation in (5.2) if $(u_0,u_1)_{L^2\times L^2} \ge 0$. We assume in what follows that $(u_0,u_1)_{L^2\times L^2} < 0$. Since $H'(t) \le 0$ when $H(t) \ge 0$, and $H'(t) < 0$ when $H(t) > 0$, we can write that if $H(t_0) < \alpha$ for some $t_0 \ge 0$, and some $\alpha > 0$, then $H(t) < \alpha$ for all $t \ge t_0$. In particular, if $H(t_1) \ge \alpha$ for some $t_1 > 0$ and some $\alpha > 0$, then $H(t) \ge \alpha$ for all $t \in [0,t_1]$. By (3.10) we have that $H''(t) \ge \varepsilon m H(t)$ for all $t \in [0,t_1]$. It follows that

$$H'(t_1) \ge H'(0) + t_1 \varepsilon m \alpha \ . \tag{5.3}$$

By Lemma 3.1 we necessarily have that $H'(t_1) < 0$ since $H(t_1) > 0$. Therefore, by (5.3), we get that $\alpha \le -H'(0)/(\varepsilon m t_1)$, and we proved that if $t = t_1$ is such that $H(t) \ge \alpha > 0$, then $\alpha \le -H'(0)/(\varepsilon m t)$. In particular, the second equation in (5.2) holds true. This ends the proof of the proposition. □

In addition to Proposition 5.1, we also get a bound for the derivative of the L^2-norm of u. By Theorem 3.1 we may assume in Lemma 5.1 that $E(u_0,u_1) > 0$. More precisely, we get that the following lemma holds true.

Lemma 5.1. *Let f satisfy (1.1) and (3.1), and let u be a nontrivial solution of (0.1) with Cauchy data u_0, u_1. If u exists on the half line $\mathbb{R}^+$, then*

$$|L_2'(t)| \le \frac{2(2+\varepsilon)E(u_0,u_1)}{\sqrt{\varepsilon m(4+\varepsilon)}} + \left(1 - \chi_{\{t\ge t_0\}}\right) L_2'(0)^- \tag{5.4}$$

for all $t \ge 0$, where

$$t_0 = \frac{\sqrt{\varepsilon m(4+\varepsilon)}\|u_0\|_{L_2}^2}{2(2+\varepsilon)E(u_0,u_1)} \ , \tag{5.5}$$

L_2 is as in (3.5), E is given by (1.3), and $\chi_{\{t\ge t_0\}}$ is the characteristic function of the interval $[t_0,+\infty)$.

Proof. We let Φ be the function given by (3.20) in the proof of Theorem 3.1, and let $\Psi = \Phi - L_2'$. By (3.22) in the proof of Theorem 3.1,

$$\Phi'(t) \ge \sqrt{\varepsilon m(4+\varepsilon)}\Phi(t) \tag{5.6}$$

for all $t \geq 0$. Assuming that there exists $t_1 \geq 0$ such that $\Phi(t_1) > 0$, we get with Gronwall's inequality that

$$\Phi(t) \geq \Phi(t_1)e^{\sqrt{\varepsilon m(4+\varepsilon)}(t-t_1)} \tag{5.7}$$

for all $t \geq t_1$. In particular, $L_2'(t) \to +\infty$ as $t \to +\infty$, and for $t \gg 1$ sufficiently large we get that $L_2(t) \gg 1$ and $L_2'(t) > 0$. By Lemma 3.1 this is impossible. In particular $\Phi(t) \leq 0$ for all $t \geq 0$. By (3.22) in the proof of Theorem 3.1 we also have that

$$\Psi'(t) \leq -\sqrt{\varepsilon m(4+\varepsilon)}\Psi(t) \tag{5.8}$$

for all $t \geq 0$. It follows form (5.8) and Gronwall's inequality that

$$\Psi(t) \leq \Psi(0)e^{-\sqrt{\varepsilon m(4+\varepsilon)}t} \tag{5.9}$$

for all $t \geq 0$, and we get with (5.9) that $\Psi(t) \leq \max(\Psi(0), 0)$ for all $t \geq 0$. Since we also have that $\Phi(t) \leq 0$ for all $t \geq 0$, it follows that

$$\begin{aligned} |L_2'(t)| &\leq \max\left(-L_2'(0), \frac{2(2+\varepsilon)E(u_0,u_1)}{\sqrt{\varepsilon m(4+\varepsilon)}}\right) \\ &\leq \frac{2(2+\varepsilon)E(u_0,u_1)}{\sqrt{\varepsilon m(4+\varepsilon)}} + L_2'(0)^- \end{aligned} \tag{5.10}$$

for all $t \geq 0$. Let now t_0 be as in (5.5). By (5.8) and Gronwall's inequality we can write that

$$\Psi(t) \geq \Psi(t_0)e^{\sqrt{\varepsilon m(4+\varepsilon)}(t_0-t)} \tag{5.11}$$

for all $0 \leq t \leq t_0$. If $\Psi(t_0) > 0$ we get with (5.11) that $\Psi(t) > 0$ for all $0 \leq t \leq t_0$. In particular,

$$L_2'(t) < \frac{-2(2+\varepsilon)E(u_0,u_1)}{\sqrt{\varepsilon m(4+\varepsilon)}} \tag{5.12}$$

for all $0 \leq t \leq t_0$, and by integrating (5.12) on $[0, t_0]$, we get with (5.5) that $L_2(t_0) < 0$. Since, by definition, $L_2(t) \geq 0$ for all t, this is a contradiction and it follows that $\Psi(t_0) \leq 0$. By (5.8) and Gronwall's inequality we then get that $\Psi(t) \leq 0$ for all $t \geq t_0$. Since we also have that $\Phi(t) \leq 0$ for all $t \geq 0$, it follows that

$$|L_2'(t)| \leq \frac{2(2+\varepsilon)E(u_0,u_1)}{\sqrt{\varepsilon m(4+\varepsilon)}} \tag{5.13}$$

for all $t \geq t_0$. We get (5.4) by combining (5.10) and (5.13). This ends the proof of Lemma 5.1. □

We prove in the sequel that the uniform L_2-bounds of Proposition 5.1 extend to bounds on the whole kinetic energy E_0 for particular values of p in (1.1). Such bounds for Klein-Gordon equations in domains of the Euclidean space were first proved by Cazenave [8] with a nonlinear term growing at most like half the critical Sobolev exponent for H^1-embeddings. The kinetic energy associated to Klein-Gordon equations controls the H^1-norm in $1 + n$ dimensions. For (0.1) we may regard the spatial Sobolev space H^2 as a subset of the spatial Sobolev space H^1 and then restrict ourselves to controlling the sole H^1-norm in $1 + n$ dimensions. This loss of control on the derivatives when passing from H^2 to H^1 has no cost when $n = 1$, a dimension where we still get with this approach the full range $p > 1$ in (1.1), but it has a cost when $n \geq 2$ by imposing a condition like $p \leq 3$ in (1.1). Lemma 5.2 below allows us to recover the full range of exponents when $n = 2$, namely $p > 1$ arbitrary in (1.1) when $n = 2$. It also enables us, see (5.21) below, to get better exponents than 3 when $n = 3, 4$, and to get better exponents than $2^\sharp/2$ when $n \geq 5$.

Lemma 5.2. *Let $I \subset \mathbb{R}$ be an interval and $u \in H^1(I, L^2) \cap L^2(I, H^2)$ be such that*

$$\int_I E_0(u(s), u_t(s))ds = N^2 \tag{5.14}$$

for some $N \geq 0$. Then

$$\begin{aligned} &\|u\|^2_{L^\infty(I,L^2)} \leq 2\left(\frac{1}{m|I|} + \frac{2}{\sqrt{m}}\right) N^2 \text{ and} \\ &\|\nabla u\|^2_{L^\infty(I,L^2)} \leq 2\left(2 + \frac{1}{|I|\sqrt{m}}\right) N^2 , \end{aligned} \tag{5.15}$$

where $|I| \leq +\infty$ is the length of I.

Proof. Without loss of generality we may assume that I is bounded. First, we suppose that $u \in C^1(I, H^2)$. Since N is finite, by the mean value theorem, there exists some time $t_0 \in I$ such that

$$E_0(u(t_0), u_t(t_0)) = \frac{N^2}{|I|} ,$$

where N is as in (5.14). Then,

$$\begin{aligned} &\int_{\mathbb{R}^n} |u(t_0, y)|^2 dy \leq \frac{2N^2}{m|I|} \text{ and} \\ &\int_{\mathbb{R}^n} |\nabla u(t_0, y)|^2 dy = -\int_{\mathbb{R}^n} u(t_0, y)\Delta u(t_0, y)dy \leq \frac{2N^2}{\sqrt{m}|I|}. \end{aligned} \tag{5.16}$$

Now, let $t \in I$. By time symmetry, we can suppose $t_0 \leq t$. We bound the gradient norm by writing that

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla u(t, y)|^2 dy &= \int_{\mathbb{R}^n} \left(|\nabla u(t_0, y)|^2 + 2\int_{t_0}^t \nabla u_t(s, y)\nabla u(s, y)ds\right) dy \\ &\leq \int_{\mathbb{R}^n} |\nabla u(t_0, y)|^2 dy - 2\int_{t_0}^t \int_{\mathbb{R}^n} u_t(s, y)\Delta u(s, y)dyds \\ &\leq 2\left(2 + \frac{1}{\sqrt{m}|I|}\right) N^2 , \end{aligned} \tag{5.17}$$

where we have used (5.16) in the last inequality. We bound the L^2 norm similarly by writing that

$$\begin{aligned} \int_{\mathbb{R}^n} |u(t, y)|^2 dy &= \int_{\mathbb{R}^n} \left(|u(t_0, y)|^2 + 2\int_{t_0}^t u_t(s, y)u(s, y)ds\right) dy \\ &\leq 2\left(\frac{1}{m|I|} + \frac{2}{\sqrt{m}}\right) N^2 , \end{aligned} \tag{5.18}$$

where, again, we have used (5.16) in the last inequality. Then (5.15) follows from (5.17) and (5.18) in case $u \in C^1(I, H^2)$. In case $u \in H^1(I, L^2) \cap L^2(I, H^2)$, the result follows by density. □

Corollary 5.1. *Let $n \geq 3$, $I \subset \mathbb{R}$ be an interval such that $|I| \geq 1$, and (a, b) satisfy* (1.44) *with $a \geq 6$ if $n = 3, 4$, and $a \geq 2$ if $n \geq 5$. Let $u \in H^1(I, L^2) \cap L^2(I, H^2)$ satisfy* (5.14)*. Then*

$$\|u\|_{L^a(I,L^b)} \leq CN \ , \tag{5.19}$$

where C does not depend on u, a and I.

Proof. Assuming that $n \geq 5$, we get from Lemma 5.2 that the bound holds true for the endpoints $(a, b) = (\infty, 2^\star)$ and $(a, b) = (2, 2^\sharp)$. Then (5.19) follows by interpolation. Applying the same strategy when $n = 3, 4$, we only have to prove (5.19) in the case $a = 6$. In this case, for $u \in H^2$,

$$\|u\|_{L^{\frac{6n}{3n-8}}} \leq C\|u\|_{H^{\frac{4}{3}}} \leq C\|u\|_{H^1}^{\frac{2}{3}}\|u\|_{H^2}^{\frac{1}{3}} \tag{5.20}$$

and (5.19) follows from (5.14), (5.15) and (5.20). This neds the proof of the corollary. □

For $n \geq 3$, we define p_n by

$$p_n = \frac{n+6}{n-2}. \tag{5.21}$$

As is easily checked,

$$p_n = 2^\sharp - 1 + O(\frac{1}{n^2})$$

as $n \to +\infty$, and $p_n \geq 2^\sharp/2$ when $n \geq 6$ with equality if and only if $n = 6$. Theorem 5.1 states as follows. Any $p > 1$ in (1.1) is allowed in the theorem when $n = 1, 2$.

Theorem 5.1. *Let f satisfy (1.1) and (3.1) with $p = p_n$ when $n \geq 3$, where p_n is as in (5.21). There exists $K \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ with $K(0) = 0$ such that if u is a solution of (0.1) with Cauchy data u_0, u_1, and if u exists on the half line $\mathbb{R}^+$, then*

$$E_0\left(u(t), u_t(t)\right) \leq K\left(E_0(u_0, u_1)\right) \tag{5.22}$$

for all $t \geq 0$, where E_0 is given by (1.3). Moreover, there is a time $t_0 \geq 0$ such that $E_0\left(u(t), u_t(t)\right) \leq K\left(E(u_0, u_1)\right)$ for all $t \geq t_0$, where E is given by (1.3).

Proof. First we prove the theorem assuming that $n = 1, 2$ and p is arbitrary. By (3.10) we can write that

$$L_2''(t) \geq 2\varepsilon E_0(u, u_t) - 2(2+\varepsilon)E(u_0, u_1)$$

for all $t \geq 0$, where L_2 is as in (3.5). Let t_0 be as in (5.5). By Lemma 5.1 and the above inequality we get that

$$\begin{aligned}\int_{t_1}^{t_2} E_0\left(u(s), u_t(s)\right) ds &\leq \frac{1}{2\varepsilon}\left(L_2'(t_2) - L_2'(t_1)\right) \\ &\quad + \left(1 + \frac{2}{\varepsilon}\right) E(u_0, u_1)(t_2 - t_1) \\ &\leq \frac{2(2+\varepsilon)E(u_0, u_1)}{\varepsilon\sqrt{\varepsilon m(4+\varepsilon)}} + \frac{1}{\varepsilon}\left(1 - \chi_{\{t_1 \geq t_0\}}\right)|L_2'(0)| \\ &\quad + \left(1 + \frac{2}{\varepsilon}\right) E(u_0, u_1)(t_2 - t_1)\end{aligned} \tag{5.23}$$

for all $t_1 \le t_2$. Let $h \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ be given by $h(X) = \Lambda X(1 + X^{(p-1)/2})$ for $X \ge 0$, where $\Lambda = C_2$ is as in (4.2), and $S \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ be given by

$$S(X) = \left(1 + \frac{2}{\varepsilon}\right)\left(1 + \frac{2}{\sqrt{\varepsilon m(4+\varepsilon)}}\right)h(X) + \frac{2(m+1)}{\varepsilon m}X \tag{5.24}$$

for $X \ge 0$. As is easiliy checked, $S(0) = 0$. Moreover, by (3.7), (4.2), and (5.23) we can write that

$$\begin{aligned} &\int_t^{t+1} E_0\left(u(s), u_t(s)\right) ds \le S\left(E_0(u_0, u_1)\right) \text{ for all } t \ge 0 \text{ , and} \\ &\int_t^{t+1} E_0\left(u(s), u_t(s)\right) ds \le S\left(E(u_0, u_1)\right) \text{ for all } t \ge t_0 \text{ .} \end{aligned} \tag{5.25}$$

Now, if $p+1 \le 2^\star$, conservation of energy, (5.15) and (5.25) and Sobolev's inequality ensures that $E_0(u, u_t)$ remains bounded since

$$\begin{aligned} E_0(u(t), u_t(t)) &= E(u(t), u_t(t)) + \int_{\mathbb{R}^n} F(u(t))dx \\ &\le E(u_0, u_1) + C\left(\|u(t)\|_{L^2}^2 + \|u(t)\|_{H^1}^{p+1}\right) \\ &\le E(u_0, u_1) + C\left(\Gamma + \Gamma^{\frac{p+1}{2}}\right), \end{aligned} \tag{5.26}$$

where Γ stands for $S\left(E_0(u_0, u_1)\right)$ if $t < t_0$, and $S\left(E(u_0, u_1)\right)$ if $t \ge t_0$. This settles the cases $n = 1, 2$. At this point it remains to treat the limit case when $p = (n+6)/(n-2)$ and $n \ge 3$. We treat first the case of high dimensions $n \ge 5$, in which case, we can rely on the Strichartz estimate of Lemma 1.1. We let (a, b) be the S-admissible pair defined by $a = 2$, $b = 2^\star$. We let also $q = 2p$ and define r, c to be such that (q, r) is B-admissible and (q, c) is B-intermediate in the sense of (1.44). By Corollary 5.1, and by (5.25), we can write that

$$\left(\int_t^{t+1} \|u\|_{L^c}^q ds\right)^{1/q} \le C\sqrt{\Gamma} \tag{5.27}$$

where $C > 0$ does not depend on t. Let us now fix $t \ge 1$. By (5.25) there exists $t_1 < t < t_1 + 1$ such that

$$E_0(u(t_1), u_t(t_1)) \le \Gamma, \tag{5.28}$$

where Γ is defined as in (5.26). Let h_1 and h_2 be as in (1.17) in Section 1. By the linear theory in Lemma 1.1 of Section 1, since h_1 is Lipschitz and h_2 satisfy (1.17), we can write that for any interval $I = [\theta, \theta']$ of length less than 1,

$$\begin{aligned} \|u\|_{L^q(I, L^r)} &\le C\left(\sqrt{\Lambda(\theta)} + \|h_1(u)\|_{L^1(I, L^2)} + \|h_2(u)\|_{L^{a'}(I, L^{b'})}\right) \\ &\le C_s\left(\sqrt{\Lambda(\theta)} + \|u\|_{L^1(I, L^2)} + \|u\|_{L^{pa'}(I, L^{pb'})}^p\right) \end{aligned} \tag{5.29}$$

where $\Lambda(t) = E_0\left(u(t), u_t(t)\right)$, and $C, C_s > 0$ do not depend on t and j. Now we let $t_2 = t < t_3 < \cdots < t_{k+1} = t_1 + 1$ be a partition of $[t_1, t_1 + 1]$ such that

$$\frac{1}{4C_s} \le \|u\|_{L^q([t_i, t_{i+1}], L^c)}^{p-1} \le \frac{1}{2C_s}$$

for $i = 1, \ldots, k$, where C_s is the constant appearing in inequality (5.29). Then

$$k \le \left((4C_s)^{\frac{1}{p-1}} C\sqrt{\Gamma}\right)^q \text{ .}$$

By Hölder's inequality, we get that

$$\|u\|^p_{L^{pa'}(I,L^{pb'})} \le \|u\|^{p-1}_{L^{pa'}(I,L^c)}\|u\|_{L^{pa'}(I,L^r)} \tag{5.30}$$

for any bounded interval $I \subset \mathbb{R}$. Noting that $q = pa'$, it follows from (5.25), (5.29) and (5.30) that, for any j,

$$\begin{aligned}\|u\|_{L^q([t_j,t_{j+1}],L^r)} &\le C_s\left(\sqrt{\Lambda(t_j)} + \sqrt{\frac{2}{m}\Gamma} + \|u\|^{p-1}_{L^q([t_j,t_{j+1}],L^c)}\|u\|_{L^q([t_j,t_{j+1}],L^r)}\right)\\ &\le C_s\left(\sqrt{\Lambda(t_j)} + \sqrt{\frac{2}{m}\Gamma} + \frac{1}{2C_s}\|u\|_{L^q([t_j,t_{j+1}],L^r)}\right)\\ &\le C\left(\sqrt{\Lambda(t_j)} + \sqrt{\frac{2}{m}\Gamma}\right)\end{aligned} \tag{5.31}$$

where $C > 0$ does not depend on j and t. Applying the Strichartz estimates again, it follows from (5.31) that

$$\begin{aligned}\sup_{s\in[t_j,t_{j+1}]}\sqrt{\Lambda(s)} &\le C\left(\sqrt{\Lambda(t_j)} + \sqrt{\frac{2}{m}\Gamma} + \|u\|^{p-1}_{L^q([t_j,t_{j+1}],L^c)}\|u\|_{L^q([t_j,t_{j+1}],L^r)}\right)\\ &\le C\left(\sqrt{\Lambda(t_j)} + \sqrt{\frac{2}{m}\Gamma}\right)\end{aligned} \tag{5.32}$$

where, again, $C > 0$ in (5.32) does not depend on j. In particular,

$$\sqrt{\Lambda(t_{j+1})} \le C\left(\sqrt{\Lambda(t_j)} + \sqrt{\frac{2}{m}\Gamma}\right)$$

and, as a consequence, we get that

$$\begin{aligned}\sqrt{\Lambda(t)} &\le \sup_{s\in[t_1,t_1+1]}\sqrt{\Lambda(s)}\\ &\le C\Gamma^{\frac{q+1}{2}}\ .\end{aligned} \tag{5.33}$$

In particular, we get with (5.33) that the theorem holds true when $p = (n+6)/(n-2)$ and $n \ge 5$. Now, at this point, we assume that $n = 3, 4$ and we use the following Strichartz estimates that can be deduced from the original one in much the same way as the intermediate Strichartz estimates in (1.45). Let $u \in C^0([0,T],H^2) \cap C^1([0,T],L^2)$ be a strong solution of (1.7) where $T \le 1$ and $k \in C^0([0,T],H^{-2})$. We let also (q,r) be such that $2 \le q, r < \infty$ and

$$\frac{2}{q} + \frac{n}{r} = \frac{2n-5}{4}$$

and (a,b) be such that $2 \le a, b < \infty$ and

$$\frac{2}{a} + \frac{n}{b} = \frac{2n-3}{4}\ .$$

Then there exists $C > 0$ independent of u such that

$$\|u\|_{L^q([0,T],L^r)} \le C\left(\sqrt{E_0(u_0,u_1)} + \|k_1\|_{L^1([0,T],L^2)} + \|k_2\|_{L^{a'}([0,T],L^{b'})}\right) \tag{5.34}$$

for every decomposition $k = k_1 + k_2$. In order to prove (5.34) we write that $u = u_1 + u_2$, where u_1 satisfies (1.7) with initial data (u_0, u_1) and $k = k_1$, and u_2 satisfies (1.7) with zero initial data and $k = k_2$. On what concerns u_1 we can use the Strichartz estimates (1.8), while for u_2 we proceed in the same way we did when proving (1.45), except that we replace (1.46) by

$$\begin{aligned} &\|v\|_{L^\infty([0,T],B_{2,2}^{-\frac{3}{4}})} \le C\|k\|_{L^{a'}([0,T],B_{c',2}^{-1})} \quad \text{and} \\ &\|v\|_{L^2([0,T],B_{2^\star,2}^{-\frac{3}{4}})} \le C\|k\|_{L^{a'}([0,T],B_{c',2}^{-1})} , \end{aligned} \tag{5.35}$$

where again $c > 1$ is such that (a, c) is S-admissible. Now, we proceed as in the case $n \ge 5$. We find $t_1 \in [t-1, t]$ such that (5.28) holds true. Then, we split $[t_1, t_1+1]$ into disjoint intervals $[t_i, t_{i+1}]$. On each interval $[t_i, t_{i+1}]$ we use the Strichartz estimates (5.34) as above, with $q = r = 4(n+2)/(2n-5)$ and $a = b = 4(n+2)/(2n-3)$. We get that

$$\begin{aligned} &\|u\|_{L^q([t_i,t_{i+1}]\times\mathbb{R}^n)} \\ &\le C\left(\sqrt{\Lambda(t_i)} + \|h_1(u)\|_{L^1([t_i,t_{i+1}],L^2)} + \|h_2(u)\|_{L^{a'}([t_i,t_{i+1}]\times\mathbb{R}^n)}\right) \\ &\le C\left(\sqrt{\Lambda(t_i)} + \sqrt{\Gamma}\right) + C^s\|u^p\|_{L^{a'}([t_i,t_{i+1}]\times\mathbb{R}^n)} \\ &\le C\left(\sqrt{\Lambda(t_i)} + \sqrt{\Gamma}\right) + C^s\|u\|^{p-1}_{L^{\frac{2(n+2)}{n-2}}([t_i,t_{i+1}]\times\mathbb{R}^n)}\|u\|_{L^q([t_i,t_{i+1}]\times\mathbb{R}^n)} . \end{aligned} \tag{5.36}$$

Now we let the t_i's be such that

$$\frac{1}{4C^s} \le \|u\|^{p-1}_{L^{\frac{2(n+2)}{n-2}}([t_i,t_{i+1}]\times\mathbb{R}^n)} \le \frac{1}{2C^s} ,$$

where C^s is the constant appearing in (5.36). By Corollary 5.1 we have that

$$k \le (4C^s)^{\frac{n+2}{4}}(C\Gamma)^{\frac{n+2}{n-2}} ,$$

and by (5.36),

$$\|u\|_{L^q([t_i,t_{i+1}]\times\mathbb{R}^n)} \le C\left(\sqrt{\Lambda(t_i)} + \sqrt{\Gamma}\right) .$$

Now, thanks to the Strichartz estimates of Lemma 1.1, we get that

$$\begin{aligned} &\|u\|_{L^\infty([t_i,t_{i+1}],H^2)} + \|u_t\|_{L^\infty([t_i,t_{i+1}],L^2)} \\ &\le C\left(\sqrt{\Lambda(t_i)} + \|h_1(u)\|_{L^1([t_i,t_{i+1}],L^2)} + \|h_2(u)\|_{L^{\frac{2(n+2)}{n+4}}([t_i,t_{i+1}]\times\mathbb{R}^n)}\right) \\ &\le C\left(\sqrt{\Lambda(t_i)} + \sqrt{\Gamma} + \|u\|^{p\theta}_{L^q([t_i,t_{i+1}]\times\mathbb{R}^n)}\|u\|^{p(1-\theta)}_{L^2([t_i,t_{i+1}]\times\mathbb{R}^n)}\right) \\ &\le C\left(\sqrt{\Lambda(t_i)} + \sqrt{\Gamma} + (\sqrt{\Lambda(t_i)} + \sqrt{\Gamma})^p\right), \end{aligned} \tag{5.37}$$

where $\theta = 4(3n+10)/(9(n+6))$. In particular, we get a bound

$$\sqrt{\Lambda(t_{i+1})} \le G(\sqrt{\Lambda(t_i)}, \sqrt{\Gamma})$$

and iterating this a finite number of times, we cover $[t_1, t_1+1]$ and find a uniform bound for $E_0(u(t), u_t(t))$ depending only on Γ. This finishes the proof of Theorem 5.1 in case $n = 3, 4$. Theorem 5.1 is proved. □

As a remark, the time t_0 in Theorem 5.1 is actually $1 + t_0$, where t_0 is as in (5.5). Another remark is that there is a simplier proof of Theorem 5.1 in case $n \geq 3$ and $p < \min(p_n, 2^\star + 1)$. Indeed, we may assume without loss of generality that $p \geq 2^\star - 1$. By conservation of the Energy we can write that given $t \geq 0$,

$$\begin{aligned} \|u\|_{H^2}^2 &\leq 2E(u_0,u_1) + 2\int_{\mathbb{R}^n} F(u)dx \\ &\leq \mu\|u\|_2^2 + C\left(1 + \|u\|_{L^{p+1}}^{p+1}\right) \\ &\leq \frac{\mu}{m}\|u\|_{H^2}^2 + C\left(1 + \|u\|_{L^{2^\star}}^{(p+1)\theta}\|u\|_{L^q}^{(p+1)(1-\theta)}\right) \\ &\leq \frac{\mu}{m}\|u\|_{H^2}^2 + C\left(1 + \|u\|_{L^{2^\star}}^{(p+1)\theta}\|u\|_{H^2}^{(p+1)(1-\theta)}\right) \end{aligned} \tag{5.38}$$

where q is taken arbitrary large if $n \leq 4$, while $q = 2^\sharp$ if $n \geq 5$, and

$$\theta = \frac{\frac{1}{p+1} - \frac{1}{q}}{\frac{1}{2^\star} - \frac{1}{q}}.$$

If $n = 3, 4$, we can take $q \to \infty$ and if $p < 2^\star + 1$, we get, for q sufficiently large, that $(1-\theta)(p+1) < 2$, while if $n \geq 5$, and if $p < (n+6)/(n-2)$, then $(1-\theta)(p+1) < 2$. In particular, the energy stays bounded, controled by some function of $\|u\|_{L^\infty(\mathbb{R},H^1)}$. This proves Theorem 5.1 in case $n \geq 3$ and $p < \min(p_n, 2^\star + 1)$

When the solution u exists on the whole line $\mathbb{R}$, Proposition 5.1, Lemma 5.1, and Theorem 5.1 can be refined. This is what we prove in Corollary 5.2 below.

Corollary 5.2. *Let f satisfy (1.1) and (3.1) with $p \leq p_n$ when $n \geq 3$, where p_n is as in (5.21). Let u be a solution of (0.1) with Cauchy data u_0, u_1. If u exists on the whole line $\mathbb{R}$, then*

$$\begin{aligned} L_2(t) &\leq \frac{2(2+\varepsilon)}{\varepsilon m}E(u_0,u_1)\ , \ \textit{and} \\ |L_2'(t)| &\leq \frac{2(2+\varepsilon)E(u_0,u_1)}{\sqrt{\varepsilon m(4+\varepsilon)}} \end{aligned} \tag{5.39}$$

for all $t \in \mathbb{R}$, where L_2 is as in (3.5), and E is given by (1.3). Moreover, $E_0(u(t), u_t(t)) \leq K(E(u_0,u_1))$, where E_0 is given by (1.3), and $K \in C^1(\mathbb{R}^+,\mathbb{R}^+)$ is as in Theorem 5.1.

Proof. We assume that $u \not\equiv 0$. Then, by Theorem 3.1, $E(u_0,u_1) > 0$. Let $t_0 \in \mathbb{R}$ be any point in $\mathbb{R}$. Let $\tilde{u}$ and $\hat{u}$ be given by $\tilde{u}(t) = u(t_0 + t)$ and $\hat{u}(t) = u(t_0 - t)$. Both $\tilde{u}$ and $\hat{u}$ are solutions of (0.1) defined on the half line $t \geq 0$. Let $\tilde{H}$ and $\hat{H}$ be the corresponding H-functions given by (3.6). If H is the H-function in (3.6) with respect to u, we get with the conservation of the total energy in Theorem 1.1 that $\tilde{H}(t) = H(t_0 + t)$ and $\hat{H}(t) = H(t_0 - t)$ for all $t \geq 0$. By Lemma 3.1, $\tilde{H}'(0) \leq 0$ if $\tilde{H}(0) \geq 0$, and $\hat{H}'(0) \leq 0$ if $\hat{H}(0) \geq 0$. It follows that $H'(t_0) = 0$ if $H(t_0) \geq 0$. By (3.10), $H'' \geq \varepsilon m H$. This clearly implies that $H(t) \leq 0$ for all $t \in \mathbb{R}$. In particular, the first equation in (5.39) is proved. Moreover, if t_0 is as in (5.5), then

$$t_0 \leq \sqrt{\frac{4+\varepsilon}{\varepsilon m}}\ . \tag{5.40}$$

Let t_1 be larger than the right hand side in (5.40), and let t_2 be any real number such that $t_2 \geq t_1 + 1$. Let also $\tilde{u}$ be given by $\tilde{u}(t) = u(t - t_2)$. Then $\tilde{u}$ solves (0.1) with Cauchy data $u(-t_2)$, $u_t(-t_2)$. We apply Lemma 5.1 and Theorem 5.1 to $\tilde{u}$. By the first equation in (5.39), which gives that the t_0 for $\tilde{u}$ is also bounded from above as in (5.40), and by the conservation of the total energy in Theorem 1.1, we get that the second equation in (5.39) and the bound $E_0\left(u(t),u_t(t)\right) \le K\left(E(u_0,u_1)\right)$ hold true for $t \ge 1+t_1-t_2$. Since $t_2 > 1+t_1$ can be chosen arbitrarily large, it follows that the second equation in (5.39) and the bound $E_0\left(u(t),u_t(t)\right) \le K\left(E(u_0,u_1)\right)$ hold true for all $t \in \mathbb{R}$. This ends the proof of the corollary. □

The second equation in (5.39) could also have been proved by coming back to Theorem 3.1. The equation is indeed a direct consequence of Theorem 3.1 and of the remark that both $t \to u(t)$ and $t \to u(-t)$ exist on the half line $t \ge 0$. Another corollary to Theorem 5.1, and more precisely to the proof of Theorem 5.1, is as follows.

Corollary 5.3. *Let f satisfy (1.1) and (3.1) with $p \le p_n$ when $n \ge 3$, where p_n is as in (5.21). Let u be a solution of (0.1) with Cauchy data u_0, u_1. Then u exists on the whole line $\mathbb{R}$ if and only if*

$$\left|(u,u_t)_{L^2\times L^2}\right| \le \frac{2+\varepsilon}{\sqrt{\varepsilon m(4+\varepsilon)}} E(u_0,u_1) \tag{5.41}$$

for all t, where $(u,u_t)_{L^2\times L^2}$ is the L^2-scalar product of u with u_t.

Proof. If u exists on the whole line $\mathbb{R}$, then (5.41) holds true by Corollary 5.2. Conversely, we suppose that (5.41) holds true for all t where u is defined. By (3.10) we can write that

$$L_2''(t) \ge 2\varepsilon E_0(u,u_t) - 2(2+\varepsilon)E(u_0,u_1) \tag{5.42}$$

for all $t \ge 0$, where L_2 is as in (3.5). By (5.41) and (5.42) we then get that

$$\begin{aligned}
&\int_{t_1}^{t_2} E_0\left(u(s),u_t(s)\right) ds \\
&\le \frac{1}{2\varepsilon}\left(L_2'(t_2) - L_2'(t_1)\right) + \left(1+\frac{2}{\varepsilon}\right) E(u_0,u_1)(t_2-t_1) \\
&\le \frac{2(2+\varepsilon)E(u_0,u_1)}{\varepsilon\sqrt{\varepsilon m(4+\varepsilon)}} + \left(1+\frac{2}{\varepsilon}\right) E(u_0,u_1)(t_2-t_1) \\
&\le \frac{2(2+\varepsilon)E(u_0,u_1)}{\varepsilon\sqrt{\varepsilon m(4+\varepsilon)}} + \left(1+\frac{2}{\varepsilon}\right) E(u_0,u_1)
\end{aligned} \tag{5.43}$$

for all $0 \le t_1 \le t_2$ such that u exists on $[0,t_2]$ and $t_2 \le t_1+1$. The arguments developed in the proof of Theorem 5.1 together with (5.43), letting S be a constant function in (5.25), give that $E_0(u,u_t)$ is bounded for $t \ge 0$. By Proposition 2.2 we then get that u exists for all $t \ge 0$. By reversing time, letting $\tilde{u}(t) = u(-t)$ for $t \le 0$, we also get that u exists for all $t \le 0$, and hence that u exists on the whole line $\mathbb{R}$. This proves the corollary. □

6. H^4 SOLUTIONS

In this section, we investigate the case where the initial data have more regularity. We focus on the case where the initial data is in $H^4 \times H^2$ instead of $H^2 \times L^2$ and prove that the corresponding solution has itself more regularity as well. A preliminary lemma we need is as follows.

Lemma 6.1. *Let $T \le 1$ and $k \in C^0([0,T],L^2)$. Assume that for some derivative $\partial = \partial_\alpha$, where $\alpha = t$ or $\alpha = 1,\dots,n$, $\partial k \in L^{a'}([0,T],L^{b'})$ for some S-admissible pair (a,b). Let $w \in C^0([0,T],H^2) \cap C^1([0,T],L^2) \cap C^2([0,T],H^{-2})$ be a strong solution of equation* (1.7) *with Cauchy data $(w_0,w_1) \in H^4 \times H^2$. Then there holds that $\partial w \in C^0([0,T],H^2) \cap C^1([0,T],L^2) \cap C^2([0,T],H^{-2})$, and*

$$\begin{aligned} &\|\partial w\|_{C^0([0,T],H^2)} + \|\partial w\|_{L^c([0,T],L^d)} \\ &\le C\left(\|w_0\|_{H^4} + \|w_1\|_{H^2} + \|k(0)\|_{L^2} + \|\partial k\|_{L^{a'}([0,T],L^{b'})}\right) \end{aligned} \tag{6.1}$$

where (c,d) is any B-admissible pair, and $C > 0$ is independent of T, w and k. Besides, in case $\partial = \partial_t$, it also holds that w enjoys the additional regularity that $w \in C^0([0,T],H^4) \cap C^1([0,T],H^2) \cap C^2([0,T],L^2)$, and

$$\|w\|_{C^0([0,T],H^4)} \le C\left(\|w_0\|_{H^4} + \|w_1\|_{H^2} + \|k\|_{C^0([0,T],L^2)} + \|k_t\|_{L^{a'}([0,T],L^{b'})}\right), \tag{6.2}$$

where $C > 0$ is independent of T, w and k.

Proof. By density, it suffices to prove this for smooth functions $w_0, w_1 \in C_0^\infty(\mathbb{R}^n)$ and $h \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$. First, we treat the case when $\partial = \partial_t$. We let $v = \partial_t w$. Then v satisfies (1.7) with k_t instead of k, $v(0) = w_1$, and $v_t(0) = k(0) - \Delta^2 w_0$. Applying the Strichartz estimates (1.8), we obtain

$$\begin{aligned} &\|v\|_{C^0([0,T],H^2)} + \|v_t\|_{C^0([0,T],L^2)} + \|v\|_{L^c([0,T],L^d)} \\ &\le C\left(\|w_1\|_{H^2} + \|w_0\|_{H^4} + \|k(0)\|_{L^2} + \|k_t\|_{L^{a'}([0,T],L^{b'})}\right) \end{aligned} \tag{6.3}$$

where $C > 0$ is independent of w, T and k. Then, using equation (1.7), we get that $\Delta^2 w = k - w_{tt} = k - v_t \in C^0([0,T],L^2)$. Besides, the Strichartz estimates (1.8) applied to w give that

$$\|w\|_{C^0([0,T],H^2)} \le C\left(\|w_0\|_{H^2} + \|w_1\|_{L^2} + \|k\|_{L^1([0,T],L^2)}\right). \tag{6.4}$$

Consequently, $w \in C^0([0,T],H^4) \cap C^1([0,T],H^2) \cap C^2([0,T],L^2)$, and (6.2) follows from (6.3) and (6.4). Now, in case $\partial = \partial_i$ for some $i = 1,\dots,n$, letting again $v = \partial w$, we get that v satisfies (1.7) with ∂k instead of k and $(v(0),v_t(0)) = (\partial w_0, \partial w_1) \in H^2 \times L^2$. Applying the Strichartz estimates (1.8), we obtain

$$\begin{aligned} &\|v\|_{C^0([0,T],H^2)} + \|v_t\|_{C^0([0,T],L^2)} + \|v\|_{L^c([0,T],L^d)} \\ &\le C\left(\|w_1\|_{H^2} + \|w_0\|_{H^4} + \|\partial k\|_{L^{a'}([0,T],L^{b'})}\right). \end{aligned} \tag{6.5}$$

Clearly, (6.5) give (6.1). This ends the proof of Lemma 6.1 □

As a remark the above developments can be easily adapted when w satisfies (1.7) with initial data $(w_0,w_1) \in H^4 \times H^2$ and $k = h_1 + h_2$, where $h_1 \in C^1([0,T],H^1)$ and $h_2 \in C^0([0,T],L^2)$ with $h_2, \partial h_2 \in L^{a'}([0,T],L^{b'})$. The main result of this section is as follows.

Proposition 6.1. *Let u be a strong solution of* (0.1) *in $[0,T]$ with f satisfying* (1.1) *and $(u_0,u_1) \in H^4 \times H^2$. Then $u \in C^0([0,T],H^4) \cap C^1([0,T],H^2) \cap C^2([0,T],L^2)$.*

Proof. We define the sequence $(u^k)_k$ by $u^0 = 0$, and $u^{k+1} = \chi(u^k)$, where χ is the contraction given in (1.23) in Section 1. We know that there exists $T' > 0$ such that $u^k \to u$ in $C^0([0,T'],H^2) \cap C^1([0,T'],L^2)$ when $n \le 4$, and such that $u^k \to u$ in $\hat{\mathcal{H}} = C^0([0,T'],H^2) \cap C^1([0,T],L^2) \cap L^q([0,T],L^r)$ when $n \ge 5$, where $q = 2(2^\sharp - 1)$

and $r = 2^\sharp(n+4)/(n+2)$. We treat the case $n \geq 5$. The case $n \leq 4$ is much simpler. We assume in what follows that $n \geq 5$, $p = 2^\sharp - 1$, and let $M > 0$ be such that for any k, the norm of u^k in $\hat{\mathcal{H}}$ is bounded by M. Given $\epsilon > 0$ to be defined later, taking T' small, $T' \leq 1$, we can assume that for any k

$$\|u^k\|_{L^q([0,T'],L^r)} \leq \epsilon. \tag{6.6}$$

We let also $R = \|u_0\|_{H^4} + \|u_1\|_{H^2}$. We first prove by induction on k that for any k, $u^k \in C^0([0,T'],H^4) \cap C^1([0,T'],H^2) \cap C^2([0,T'],L^2)$, and that for any derivative $\partial = \partial_\alpha$, $\alpha = t$ or $\alpha = 1,\dots,n$, $\partial u^k \in \hat{\mathcal{H}}$ and

$$\|\partial u^k\|_{C^0([0,T'],H^2)} + \|\partial u_t^k\|_{C^0([0,T'],L^2)} + \|\partial u^k\|_{L^q([0,T'],L^r)} \leq C(M,R)\ , \tag{6.7}$$

where $C(M,R) = C(R + R^{\frac{n+4}{n-4}} + M)$ is some constant independent of k. This is obvious when $k = 0$. Suppose our proposition holds true for some $k \geq 0$. Then $u^{k+1} = \chi(u^k)$ satisfies equation (1.7) with $k = f(u^k) - mu^k = h_1(u^k) + h_2(u^k)$, where h_1 and h_2 are the functions defined in the proof of Theorem 1.1 in Section 1. Since h_1 is lipschitz, we get that $\partial\left(h_1(u^k)\right) = h_1'(u^k)\partial u^k \in L^\infty([0,T'],L^2)$, and since h_2 satisfy (1.17), h_2 is locally lipschitz, and, using (6.6), we obtain that $\partial\left(h_2(u_k)\right) = h_2'(u^k)\partial u^k \in L^2([0,T'],L^{\frac{2n}{n+2}})$ with

$$\begin{aligned}\|h_2'(u^k)\partial u^k\|_{L^2([0,T'],L^{\frac{2n}{n+2}})} &\leq \||u^k|^{\frac{8}{n-4}}\|_{L^{\frac{n+4}{4}}([0,T'],L^{\frac{n(n+4)}{4(n+2)}})}\|\partial u^k\|_{L^q([0,T],L^r)}\\ &\leq \epsilon^{\frac{8}{n-4}}\|\partial u^k\|_{L^q([0,T'],L^r)}.\end{aligned} \tag{6.8}$$

Independently, we have that

$$\|h_1(u_0)\|_{L^2} \leq C\|u_0\|_{L^2} \text{ and } \|h_2(u_0)\|_{L^2} \leq C\|u_0\|_{H^4}^{\frac{n+4}{n-4}}. \tag{6.9}$$

Applying Lemma 6.1, estimates (6.8), (6.9), and our induction assumption, we get that $\partial u^{k+1} \in \hat{\mathcal{H}}$ and that

$$\begin{aligned}&\|\partial u^{k+1}\|_{C^0([0,T'],H^2)} + \|\partial u^{k+1}\|_{L^q([0,T'],L^r)} + \|\partial u_t^{k+1}\|_{C^0([0,T'],L^2)}\\ &\leq C\left(R + R^{\frac{n+4}{n-4}}\right) + CT'\|\partial u^k\|_{L^\infty([0,T'],L^2)} + C\epsilon^{\frac{8}{n-4}}\|\partial u^k\|_{L^q([0,T'],L^r)}\\ &\leq C\left(R + R^{\frac{n+4}{n-4}} + M\right),\end{aligned} \tag{6.10}$$

provided that ϵ and T' are chosen sufficiently small. This proves (6.7). It follows that $(u^k)_k$ is uniformly bounded in $L^\infty([0,T'],H^3)$. Besides, when $5 \leq n \leq 12$, we have that

$$\begin{aligned}\|h_2(u^k)\|_{L^\infty([0,T'],L^2)} &\leq C\|u^k\|_{L^\infty([0,T'],L^{\frac{2(n+4)}{n-4}})}^{\frac{n+4}{n-4}}\\ &\leq C\|u^k\|_{L^\infty([0,T'],H^{\frac{4n}{n+4}})}^{\frac{n+4}{n-4}}\\ &\leq C\|u^k\|_{L^\infty([0,T'],H^3)}^{\frac{n+4}{n-4}}\\ &\leq C'\left(R,M\right).\end{aligned} \tag{6.11}$$

Furthermore, $h_2(u^k) \in C^0([0,T'],L^2)$. Indeed, letting, for $t,s \geq 0$, $g^k(t,s)(x) = 0$ if $u^k(t,x) = u^k(s,x)$, and

$$g^k(t,s) = \frac{h_2(u^k(t)) - h_2(u^k(s))}{u^k(t) - u^k(s)}$$

otherwise, we get that, for $t, \theta \geq 0$,

$$h_2(u^k(t+\theta)) - h_2(u^k(t)) = (u^k(t+\theta) - u^k(t))g^k(t+\theta, t),$$

and consequently, by (1.17),

$$\begin{aligned}
&\|h_2(u^k(t+\theta)) - h_2(u^k(t))\|_{L^2} \\
&\leq \|u^k(t+\theta) - u^k(t)\|_{L^{\frac{2(n+4)}{n-4}}} \|g^k(t+\theta, t)\|_{L^{\frac{n+4}{4}}} \\
&\leq C\|u^k(t+\theta) - u^k(t)\|_{H^{\frac{4n}{n+4}}} \sup_{[t,t+\theta]} \||u^k|^{\frac{8}{n-4}}\|_{L^{\frac{n+4}{4}}} \\
&\leq C\|u^k(t+\theta) - u^k(t)\|_{L^2}^{\frac{4}{n+4}} \|u^k(t+\theta) - u^k(t)\|_{H^4}^{\frac{n}{n+4}} \sup_{[t,t+\theta]} \|u^k\|_{H^4}^{\frac{8}{n-4}} \\
&\leq C\left(\sup_{[t,t+\theta]} \|u^k\|_{H^4}^{\frac{n^2+4n+32}{n^2-16}}\right) \|u^k(t+\theta) - u^k(t)\|_{L^2}^{\frac{4}{n+4}} .
\end{aligned} \tag{6.12}$$

Since $u^k \in L^\infty([0,T'], H^4) \cap C^0([0,T'], L^2)$, we get that $h_2(u^k) \in C^0([0,T'], L^2)$. This proves the above claim. Now we can apply Lemma 6.1 thanks to (6.7) and (6.12) and conclude that $u^{k+1} \in C^0([0,T'], H^4) \cap C^1([0,T'], H^2)$ with the property that

$$\|u^k\|_{L^\infty([0,T'],H^4)} \leq C\,(R, M)\,. \tag{6.13}$$

When $n \geq 13$, we proceed as follows. Namely we write that

$$\begin{aligned}
\|h_2(u^k)\|_{L^\infty([0,T'],L^2)} &\leq C\|u^k\|^{\frac{n+4}{n-4}}_{L^\infty([0,T'],L^{\frac{2(n+4)}{n-4}})} \\
&\leq C\|u^k\|^{\frac{n+4}{n-4}}_{L^\infty([0,T'],H^{\frac{4n}{n+4}})} \\
&\leq C\left(\|u^k\|^{\frac{16}{n+4}}_{L^\infty([0,T'],H^3)}\|u^k\|^{\frac{n-12}{n+4}}_{L^\infty([0,T'],H^4)}\right)^{\frac{n+4}{n-4}} \\
&\leq C'\,(R, M)\,\|u^k\|^{\frac{n-12}{n-4}}_{L^\infty([0,T'],H^4)}
\end{aligned} \tag{6.14}$$

and, as in the case where $5 \leq n \leq 12$, see (6.12), we get that $h_2(u^k) \in C^0([0,T'], L^2)$. Then we can apply Lemma 6.1. In particular,

$$u^{k+1} \in C^0([0,T'], H^4) \cap C^1([0,T'], H^2)$$

and

$$\begin{aligned}
\|u^{k+1}\|_{L^\infty([0,T'],H^4)} &\leq C'(R, M)\left(1 + \|u^k\|^{\frac{n-12}{n-4}}_{L^\infty([0,T'],H^4)}\right) \\
&\leq C'(R, M).
\end{aligned} \tag{6.15}$$

Finally, with (6.13) and (6.15), we get that for any $n \geq 5$ and for any time $t \in [0,T']$, $\left(u^k(t)\right)_k$ is bounded in H^4 uniformly in t and k. Since it converges to $u(t)$ in H^2, we get that $u(t) \in H^4$, and that $u \in L^\infty([0,T'], H^4)$. Finally, $u \in \hat{\mathcal{H}} \cap L^\infty([0,T'], H^4) \cap W^{1,q}([0,T'], L^r)$. Hence, proceeding as in (6.12), we get that $h(u) \in C^0([0,T'], L^2)$. Applying once again Lemma 6.1 we see that $u \in C^0([0,T'], H^4) \cap C^1([0,T'], H^2) \cap C^2([0,T'], L^2)$. This ends the proof of the proposition. ☐

An interesting corollary to the above developments, where we get smooth long time solutions, is as follows.

Corollary 6.1. *Let $n \le 7$. For any Cauchy data $(u_0, u_1) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$, there exists a solution $u \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ of the cubic defocusing equation* (0.1) *with $f(u) = -u^3$. Besides, this solution is unique among all finite energy solutions.*

Proof. First we prove recursively on k that for any integer $k \ge 2$, there holds that $u \in C^0(\mathbb{R}, H^{2k}) \cap C^1(\mathbb{R}, H^{2(k-1)})$. By Proposition 6.1 this holds true when $k = 2$ and, consequently, we have that $u \in L^\infty_{loc}(\mathbb{R}, L^\infty)$. Now we assume that for some $k \ge 2$, $u \in C^0(\mathbb{R}, H^{2k})$. Then, see for instance Tao [61, Appendix A], $\Delta^k u^3 \in C^0(\mathbb{R}, L^2)$. It follows, see Lemma 1.1, that for any k, $\Delta^k u \in C^0(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, L^2)$. In particular, by induction, $u \in C^0(\mathbb{R}, H^{2k}) \cap C^1(\mathbb{R}, H^{2(k-1)})$ for all k. We have that

$$\frac{\partial^2 u}{\partial t^2} = -\Delta^2 u - mu - u^3$$

so that $u_{tt} \in C^0(\mathbb{R}, H^{2k})$ for all k. By induction, using, for instance, once again Tao [61, Appendix A] for the cubic nonlinearity, it easily follows that $u \in C^{k'}(\mathbb{R}, H^{2k})$ for all k, k'. The result follows. □

As a remark, when $n = 8$, the cubic defocusing equation (0.1) with $f(u) = -u^3$ is critical and when $n \ge 9$ it is supercritical.

7. A Segal's type theorem

We aim here in proving a Segal's type theorem for (0.1), and more precisely that long time solutions of (0.1) exist when we adopt a weaker notion of solution than the one in (1.4) and the nonlinearity in (0.1) is of defocusing type. Given $f \in C^0(\mathbb{R}, \mathbb{R})$, we say that u is a weak solution of (0.1) in $\mathbb{R}^+$ with Cauchy data $u_0 \in H^2$ and $u_1 \in L^2$ if $u \in L^\infty H^2 \cap H^{1,\infty} L^2$, $f(u) \in L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^n)$, and

$$\begin{aligned} &\int_0^{+\infty} \int_{\mathbb{R}^n} u \left(\frac{\partial^2 \varphi}{\partial t^2} + \Delta^2 \varphi + m\varphi \right) dtdx \\ &= \int_0^{+\infty} \int_{\mathbb{R}^n} f(u)\varphi dtdx - \int_{\mathbb{R}^n} u_0 \varphi_t(0) dx + \int_{\mathbb{R}^n} u_1 \varphi(0) dx \end{aligned} \tag{7.1}$$

for all $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^n)$. Note that this implies that $u \in C^0(\mathbb{R}, H^1) \cap C^0_w(\mathbb{R}, H^2)$. By extension, we say that u is a weak solution of (0.1) in $\mathbb{R}$ with Cauchy data u_0 and u_1 if u is a weak solution of (0.1) in $\mathbb{R}^+$ with Cauchy data u_0 and u_1 and if, by changing t into $-t$, we get a weak solution of (0.1) in $\mathbb{R}^+$ with Cauchy data u_0 and $-u_1$. A solution of (0.1) in the sense of (1.4) is a weak solution of (0.1). A weak solution u is said to be of finite energy, or a weak finite energy solution, if we also have that $F(u) \in L^\infty L^1$ and that $E(u, u_t) \le E(u_0, u_1)$ for almost every time t, where E is as in (1.3). We prove here that the following theorem holds true. The result for the wave equation in Euclidean space goes back to Segal [52].

Theorem 7.1. *Let $f \in C^0(\mathbb{R}, \mathbb{R})$ be locally Lipschitz and such that $f(0) = 0$. Suppose that $xf(x) \le 0$ for all $x \in \mathbb{R}$. Then, for any $u_0 \in H^2$ such that $F(u_0) \in L^1$, and any $u_1 \in L^2$, there exists a weak finite energy solution of* (0.1) *in $\mathbb{R}$ with Cauchy data u_0 and u_1.*

Proof. It clearly follows from the assumption $xf(x) \le 0$ in the theorem that

$$F(x) = \mathrm{sgn}(x) \int_0^{|x|} f(\mathrm{sgn}(x)t)\, dt \le 0 \tag{7.2}$$

for all x, where $\mathrm{sgn}(x) = \pm 1$ is the sign of x. We let $(s_k)_k$ and $(t_k)_k$, to be chosen later on, be sequences of positive real numbers such that $s_k \to +\infty$ and $t_k \to +\infty$ as $k \to +\infty$. Then we define $(f_k)_k$, where the functions $f_k : \mathbb{R} \to \mathbb{R}$ are given by

$$\begin{gathered} f_k(x) = f(-s_k) \text{ if } x \leq -s_k \ , \\ f_k(x) = f(x) \text{ if } -s_k \leq x \leq t_k \ , \text{ and} \\ f_k(x) = f(t_k) \text{ if } x \geq t_k \end{gathered} \tag{7.3}$$

for all k. The f_k's are Lipschitz functions. We may then apply Theorem 1.1 and Corollary 2.1 to get the existence of a solution u_k in $\mathbb{R}^+$ of the equation

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + mu = f_k(u) \tag{7.4}$$

with Cauchy data u_0 and u_1. Let v_k be the solution of (7.4) in $\mathbb{R}^+$ with Cauchy data u_0 and $-u_1$. Let also u_k be defined by $u_k(t,\cdot) = u_k(t,\cdot)$ if $t \geq 0$, and $u_k(t,\cdot) = v_k(-t,\cdot)$ if $t \leq 0$. As is easily checked, if we still denote by u_k the map $t \to u_k(t,\cdot)$, then u_k solves (7.4) in $\mathbb{R}$ with Cauchy data u_0 and u_1. Let E_k be given by

$$E_k = \frac{1}{2}\int_{\mathbb{R}^n} \left((\Delta u_0)^2 + mu_0^2 + u_1^2\right) dx - \int_{\mathbb{R}^n} F_k(u_0)dx \ , \tag{7.5}$$

where F_k is the primitive of f_k given by $F_k(x) = \int_0^x f_k(t)dt$. The conservation of the total energy in Theorem 1.1 gives that

$$\frac{1}{2}\int_{\mathbb{R}^n} \left((\Delta u_k)^2 + mu_k^2 + u_{k,t}^2\right) dx - \int_{\mathbb{R}^n} F_k(u_k)dx = E_k \tag{7.6}$$

for all k, where $u_{k,t} = (u_k)_t$ is the partial derivative with respect to t of u_k. Now we claim that we can choose $(s_k)_k$ and $(t_k)_k$ in (7.3) such that

$$\int_{\mathbb{R}^n} F_k(u)dx \to \int_{\mathbb{R}^n} F(u)dx \tag{7.7}$$

as $k \to +\infty$, for all $u \in H^2$ such that $F(u) \in L^1$. We prove (7.7) in what follows. As a preliminary remark, we note that

$$F_k(u) \leq 0 \tag{7.8}$$

for all k and all $u \in H^2$. Suppose now that $f(x) \to -\infty$ as $x \to +\infty$. Then we can choose the sequence $(t_k)_k$ such that $f(t_k) = \max_{x\in[t_k,+\infty)} f(x)$ for all k. With such t_k's we can write that if $u \geq 0$, then $F_k(u) \geq F(u)$. If not the case, namely if $f(x)$ does not converge to $-\infty$ as $x \to +\infty$, we choose the sequence $(t_k)_k$ such that $-f(t_k) \leq t_k$ for all k. Then, when $u \geq 0$, $F_k(u) - F(u) = 0$ if $u \leq t_k$, while

$$\begin{aligned} F_k(u) - F(u) &\geq f(t_k)\,(u - t_k) \\ &\geq -t_k\,(u - t_k) \\ &\geq -u^2 \end{aligned}$$

if $u \geq t_k$. In particular, in both cases, when $f(x) \to -\infty$ as $x \to +\infty$ and when this is not the case, it follows from the above discussion that we can always choose $(t_k)_k$ such that for any $u \in H^2$, and any k,

$$F_k(u) \geq F(u) - u^2 \tag{7.9}$$

when $u \geq 0$. In a similar way, if $f(x) \to +\infty$ as $x \to -\infty$, we choose the sequence $(s_k)_k$ such that $f(-s_k)$ is the minimum of $f(x)$ for $x \in (-\infty, -s_k]$ and all k, and if $f(x)$ does not converge to $+\infty$ as $x \to -\infty$, we choose the sequence $(s_k)_k$ such that $f(-s_k) \le s_k$ for all k. Then, as above, we get that for any $u \in H^2$, and any k

$$F_k(u) \ge F(u) - u^2 \tag{7.10}$$

when $u \le 0$. Summarizing, we get with (7.8), (7.9), and (7.10) that for any $u \in H^2$, and any k,

$$F(u) - u^2 \le F_k(u) \le 0 \; . \tag{7.11}$$

Since we also have that $F_k(u) \to F(u)$ almost everywhere in $\mathbb{R}^n$, we get with (7.11) that (7.7) holds true. Now, by (7.7), and since $F(u_0) \in L^1$, we can write that $E_k \to E(u_0, u_1)$ as $k \to +\infty$, where E_k is as in (7.5) and E is the total energy as in (1.3). In particular, up to passing to a subsequence, we can assume that $E_k \le 2E(u_0, u_1)$ for all k. By (7.6) and (7.8), we can write that

$$\begin{aligned} E_0(u_k, u_{k,t}) &= E_k + \int_{\mathbb{R}^n} F_k(u_k) dx \\ &\le 2E(u_0, u_1) \end{aligned} \tag{7.12}$$

for all k and all $t \ge 0$, where E_0 is as in (1.3). By construction of the u_k's, (7.12) holds also for $t \le 0$. By (7.12), and this remark, the u_k's are bounded in $H^1([-T,T]^{1+n})$ for any $T > 0$. Up to passing to a subsequence, we may therefore assume that for any $T > 0$, $u_k \to u$ strongly in $L^2([-T,T]^{1+n})$ and almost everywhere as $k \to +\infty$. We may also assume that $\Delta u_k \rightharpoonup \Delta u$ and that $u_{k,t} \rightharpoonup u_t$ weakly in $L^2([-T,T]^{1+n})$ as $k \to +\infty$. By Fatou's lemma, we can write that for almost every t,

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\frac{1}{2} \left((\Delta u)^2 + mu^2 + u_t^2 \right) - F(u) \right) dx \\ &\quad \le \liminf_{k \to +\infty} \int_{\mathbb{R}^n} \left(\frac{1}{2} \left((\Delta u_k)^2 + mu_k^2 + u_{k,t}^2 \right) - F_k(u_k) \right) dx \; . \end{aligned} \tag{7.13}$$

Then, by (7.13), we get that $E(u, u_t) \le E(u_0, u_1)$ for almost every time t, where E is the total energy as in (1.3). This is the nonincreasing property of the energy we ask for in the definition of weak finite energy solutions. Now we claim that

$$f_k(u_k) \to f(u) \text{ in } L^1_{loc}(\mathbb{R} \times \mathbb{R}^n) \tag{7.14}$$

as $k \to +\infty$. Given $T > 0$ arbitrary, we let $K_T = [-T,T]^{1+n}$. Multiplying by u_k the equation (7.4) satisfied by the u_k's, and integrating over K_T, we also get, after some integration by parts, that

$$\begin{aligned} \int_{K_T} |u_k f_k(u_k)| dt dx &\le \int_0^T \int_{\mathbb{R}^n} |u_k f_k(u_k)| dt dx \\ &\le -\int_0^T \int_{\mathbb{R}^n} u_k f_k(u_k) dt dx \\ &\le -\int_0^T \int_{\mathbb{R}^n} u_k \left(\frac{\partial^2 u_k}{\partial t^2} + \Delta^2 u_k + m u_k \right) dx dt \\ &\le -\int_{\mathbb{R}^n} \left(u_k(T) u_{k,t}(T) - u_0 u_1 \right) dx + \int_0^T \int_{\mathbb{R}^n} u_{k,t}^2 dt dx \\ &\le 4 \left(\frac{1}{\sqrt{m}} + T \right) E(u_0, u_1) \end{aligned} \tag{7.15}$$

for all k. Now we may use Egoroff's theorem to get that (7.14) holds true. We let $\varepsilon > 0$ be given, and let $\delta_\varepsilon > 0$ to be chosen later on. Since $f_k(u_k) \to f(u)$ almost everywhere, we can write by Egoroff's theorem that there exists a measurable subset N of K_T such that $\mathrm{Vol}(N) \le \delta_\varepsilon$, where Vol stands for the euclidian volume, and such that $f_k(u_k) \to f(u)$ uniformly in $K_T \backslash N$. Using (7.15), we can write that

$$\begin{aligned}\int_N |f_k(u_k)|\,dtdx &\le \int_{N\cap\{|u_k|\le D\}} |f_k(u_k)|\,dtdx \\ &\quad + \frac{1}{D}\int_{N\cap\{|u_k|\ge D\}} |u_k f_k(u_k)|\,dtdx \\ &\le \mathrm{Vol}(N)\max_{[-D,D]}|f| + \frac{1}{D}\int_{K_T}|u_k f_k(u_k)|\,dtdx \\ &\le \delta_\varepsilon \max_{[-D,D]}|f| + \frac{4\left(\frac{1}{\sqrt{m}}+T\right)E(u_0,u_1)}{D}\end{aligned} \tag{7.16}$$

where $D > 0$ is arbitrary and k is sufficiently large such that $D \le t_k, |s_k|$. The same upper bound holds for $\int_N |f(u)|\,dtdx$ by Fatou's lemma. In particular, by (7.16),

$$\begin{aligned}&\int_{K_T} |f_k(u_k) - f(u)|\,dtdx \\ &\le \int_N |f_k(u_k)-f(u)|\,dtdx + \int_{K_T\backslash N}|f_k(u_k)-f(u)|\,dtdx \\ &\le 2\delta_\varepsilon \max_{[-D,D]}|f| + \frac{8\left(\frac{1}{\sqrt{m}}+T\right)E(u_0,u_1)}{D} \\ &\quad + \mathrm{Vol}(K_T)\,\|f_k(u_k)-f(u)\|_{L^\infty(K_T\backslash N)}\end{aligned} \tag{7.17}$$

for all $D > 0$ and all k is sufficiently large such that $D \le t_k, |s_k|$. Choosing $D > 0$ such that

$$24\left(\frac{1}{\sqrt{m}}+T\right)E(u_0,u_1) \le \varepsilon D\ ,$$

and then $\delta_\varepsilon > 0$ such that we also have that $6\delta_\varepsilon \max_{[-D,D]}|f| \le \varepsilon$, we easily get with (7.17) that for $k \gg 1$ sufficiently large,

$$\int_{K_T} |f_k(u_k) - f(u)|\,dtdx < \varepsilon\ .$$

Since $\varepsilon > 0$ is arbitrary, this proves (7.14). Now let $\varphi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$. By multiplying (7.4) by φ and by integrating over $\mathbb{R}^+ \times \mathbb{R}^n$ we can write that

$$\begin{aligned}&\int_0^{+\infty}\int_{\mathbb{R}^n} u_k\left(\frac{\partial^2\varphi}{\partial t^2} + \Delta^2\varphi + m\varphi\right)dtdx \\ &= \int_0^{+\infty}\int_{\mathbb{R}^n} f_k(u_k)\varphi dtdx - \int_{\mathbb{R}^n} u_0\varphi_t(0)dx + \int_{\mathbb{R}^n} u_1\varphi(0)dx\ ,\end{aligned} \tag{7.18}$$

and (7.1) follows from (7.14), (7.18), and the convergence of the u_k's in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^n)$. By construction, when changing t into $-t$ and u_1 into $-u_1$, we also get that (7.1) holds true. This ends the proof of the theorem. □

8. Stability

In this section, we discuss stability of smooth C_b^4 solutions of (0.1) following the approach developed in Struwe [60] for the wave and Schrödinger equations. Here we assume that $f \in C^1(\mathbb{R}, \mathbb{R})$ is such that for any $R > 0$, there exists $C = C(R) > 1$ such that

$$\begin{aligned} &-\frac{1}{C}F(w) - Cw^2 \le -F(u+w) + F(u) + f(u)w \le -CF(w) + Cw^2 \ , \\ &|f(u+w) - f(u) - f'(u)w| \le -CF(w) + Cw^2 \end{aligned} \tag{8.1}$$

for all $|u| \le R$ and all w. Also we assume that $f(0) = 0$ and that $xf(x) \le 0$ for all x in order to recover Segal's theorem. A typical nonlinearity satisfying these assumptions is the pure power nonlinearity given by $f(u) = -|u|^{p-1}u$ for $p > 1$.

Theorem 8.1. *Suppose $u \in C_b^4([0,T] \times \mathbb{R}^n) \cap C^2([0,T], H^4)$ is a classical solution of (0.1) with Cauchy data (u_0, u_1) and f satisfying (8.1). Let v be any finite energy solution of the same equation with Cauchy data $(v_0, v_1) \in H^2 \times L^2$. For $t \ge 0$, let $\tilde{w}(t) = v(t) - u(t)$. There exists constants $C_1(u), C_2(u)$ such that, for any $t \in [0,T]$,*

$$E(\tilde{w}(t), \tilde{w}_t(t)) \le C_1 e^{C_2 t} E(\tilde{w}(0), \tilde{w}_t(0)). \tag{8.2}$$

In particular, uniqueness for the Cauchy problem with Cauchy data (u_0, u_1) holds true among weak finite energy solutions.

Proof. Let $w = -\tilde{w}$. We observe that w satisfies

$$\frac{\partial^2 w}{\partial t^2} + \Delta^2 w + mw + f(u-w) - f(u) = 0 \tag{8.3}$$

in the sense of distributions. Let $Su = \Delta u + iu_t$. We split the energy into several parts by writing that

$$E(v, v_t) = E(u, u_t) - I + II \ , \tag{8.4}$$

where

$$\begin{aligned} I &= \mathrm{Re} \int_{\mathbb{R}^n} \left(Su\overline{Sw} + muw - f(u)w \right) dx \ , \text{ and} \\ II &= \int_{\mathbb{R}^n} \left(\frac{1}{2}\left(|Sw|^2 + mw^2 \right) - \left(F(u-w) - F(u) + f(u)w \right) \right) dx. \end{aligned}$$

As a remark, $t \to I(t)$ is a continuous function of t. Now, let $(\eta_k)_k$, $\eta_k \in C_0^\infty((0,t))$, be an increasing sequence of functions such that for any k, $0 \le \eta_k \le 1$ and η_k converges almost everywhere to the characteristic function of the set $[0,t]$. In the sense of measures on $[0,t]$, we have that $\eta_k' \rightharpoonup \delta_0 - \delta_t$ vaguely as $k \to +\infty$. In what follows, $\langle \cdot, \rangle_{\mathcal{D}}$ denotes the duality product of a smooth compactly supported function and a distribution in $(0,t) \times \mathbb{R}^n$. We have that

$$\begin{aligned} I(t) - I(0) &= -\lim_{k \to +\infty} \int_0^t \eta_k'(s) I(s) ds \\ &= -\lim_{k \to +\infty} \int_0^t \int_{\mathbb{R}^n} \eta_k'(s) \left(u_t w_t + \Delta u \Delta w + muw - f(u)w \right) dx ds \ . \end{aligned} \tag{8.5}$$

Now, we assume that $u = \tilde{u} \in C_0^\infty([0,T] \times \mathbb{R}^n)$. Then, we have that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} \eta_k'(s) \left(\tilde{u}_t w_t + \Delta\tilde{u}\Delta w + m\tilde{u}w - f(\tilde{u})w\right) dxds \\
&= \langle \eta_k' \tilde{u}_t, w_t \rangle_{\mathcal{D}} + \langle \eta_k' \Delta\tilde{u}, \Delta w \rangle_{\mathcal{D}} + m \langle \eta_k' \tilde{u}, w \rangle_{\mathcal{D}} - \langle \eta_k' f(\tilde{u}), w \rangle_{\mathcal{D}} \\
&= \left\langle \frac{d}{ds} (\eta_k \tilde{u}_t) - \eta_k \tilde{u}_{tt}, w_t \right\rangle_{\mathcal{D}} + \left\langle \frac{d}{ds} (\eta_k \Delta\tilde{u}) - \eta_k \Delta\tilde{u}_t, \Delta w \right\rangle_{\mathcal{D}} \\
&+ m \left\langle \frac{d}{ds} (\eta_k \tilde{u}) - \eta_k \tilde{u}_t, w \right\rangle_{\mathcal{D}} - \left\langle \frac{d}{ds} (\eta_k f(\tilde{u})) - \eta_k f'(\tilde{u})\tilde{u}_t, w \right\rangle_{\mathcal{D}} \\
&= -\left\langle \eta_k \tilde{u}_t, \left(w_{tt} + \Delta^2 w + mw - f'(\tilde{u})w\right) \right\rangle_{\mathcal{D}} \\
&- \left\langle \eta_k \left(\tilde{u}_{tt} + \Delta^2\tilde{u} + m\tilde{u} - f(\tilde{u})\right), w_t \right\rangle_{\mathcal{D}} .
\end{aligned} \tag{8.6}$$

Then, using (8.3) and (8.6), we get

$$\begin{aligned}
&-\int_0^t \int_{\mathbb{R}^n} \eta_k'(s) \left(\tilde{u}_t w_t + \Delta\tilde{u}\Delta w + m\tilde{u}w - f(\tilde{u})w\right) dxds \\
&= \int_0^t \int_{\mathbb{R}^n} \eta_k \left(\tilde{u}_t \left(f(u) - f(u-w) - f'(\tilde{u})w\right) + \left(\tilde{u}_{tt} + \Delta^2\tilde{u} + m\tilde{u} - f(\tilde{u})\right) w_t\right) dxdt .
\end{aligned}$$

Now, by density, this remains true for u instead of $\tilde{u}$, and, with (0.1) and (8.1), we find that

$$I(0) - I(t) = -\lim_{k\to+\infty} \int_0^t \eta_k \int_{\mathbb{R}^n} \left(f(u+w) - f(u) - f'(u)w\right) u_t dxdt$$

and thus that

$$\begin{aligned}
|I(0) - I(t)| &\le C \int_0^t \int_{\mathbb{R}^n} \left(-F(\tilde{w}) + \tilde{w}^2\right) dxds \\
&\le C \int_0^t E(\tilde{w}(s), \tilde{w}_t(s)) ds ,
\end{aligned} \tag{8.7}$$

where $C = C(\|u\|_{L^\infty})$ depends on u. Moreover, since

$$w \in C^0([0,T], L^2) \cap C_w^1([0,T], L^2) ,$$

then $t \mapsto \|\tilde{w}(t)\|_{L^2}^2$ is C^1, and

$$\begin{aligned}
\|\tilde{w}(t)\|_{L^2}^2 &\le \|\tilde{w}(0)\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^n} \tilde{w}_t(s)\tilde{w}(s) dxds \\
&\le \|\tilde{w}(0)\|_{L^2}^2 + C \int_0^t E\left(\tilde{w}(s), \tilde{w}_t(s)\right) ds .
\end{aligned} \tag{8.8}$$

Independently, using (8.1), we get the following minoration of II in (8.4). Namely that

$$\begin{aligned}
II(t) &\ge \int_{\mathbb{R}^n} \left(\frac{1}{2}\left(|S\tilde{w}(t)|^2 + m\tilde{w}(t)^2\right) - \frac{1}{C} F(\tilde{w}(t)) - C\tilde{w}(t)^2\right) dx \\
&\ge \frac{1}{C} E\left(\tilde{w}(t), \tilde{w}_t(t)\right) - C\|\tilde{w}(t)\|_{L^2}^2,
\end{aligned} \tag{8.9}$$

while, at time $t = 0$ we have that

$$\begin{aligned} II(0) &\leq \int_{\mathbb{R}^n} \left(\frac{1}{2} \left(|S\tilde{w}|^2 + m\tilde{w}^2 \right) - CF(\tilde{w}) + C\tilde{w}^2 \right) dx \\ &\leq CE\left(\tilde{w}(0), \tilde{w}_t(0)\right) . \end{aligned} \tag{8.10}$$

Now, since v is a finite energy solution, we obtain that

$$0 \leq E(v(0), v_t(0)) - E(v(t), v_t(t)) = -I(0) + I(t) + II(0) - II(t) . \tag{8.11}$$

Then we use (8.9), (8.11), (8.7), and (8.10) to get that

$$\begin{aligned} E(\tilde{w}(t), \tilde{w}_t(t)) &\leq CII(t) + C\|\tilde{w}(t)\|_{L^2}^2 \\ &\leq C \int_0^t E(\tilde{w}(s), \tilde{w}_t(s)) ds + CE(\tilde{w}(0), \tilde{w}_t(0)). \end{aligned} \tag{8.12}$$

An application of Gromwall's lemma provides the conclusion. This ends the proof of the theorem. □

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