Elliptic stability and unstability tables

Elliptic stability for stationary Schrödinger equations

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(Positive solutions)

Unstable situations

$(B_\alpha + R_\alpha)$-config.	$(\sum B^i_\alpha + R_\alpha)$-config.	$(u_\infty + \sum B^i_\alpha + R_\alpha)$-config.	Unbounded Energy
(S^n, g_0), $n \geq 3$ $h_\alpha \equiv \frac{n(n-2)}{4}$ (Historical)	$(S^n/G, g_0)$, $n \geq 6$ $h_\alpha \overset{C^1}{\to} \frac{n(n-2)}{4}$ (Druet-Hebey, 2004)	(S^6, g_0), $h > 6$ $h_\alpha \overset{C^1}{\to} h$, 1-Bubble, $u_\infty \not\equiv 0$ (Druet-Hebey, 2009)	$(S^n/G, g_0)$, $n \geq 6$, $h_\alpha \overset{C^1}{\to} \frac{n(n-2)}{4}$ (Druet-Hebey, 2004)
(M, g), $n \geq 4$, $Y_g > 0$, $h_\alpha \overset{C^\infty}{\to} \frac{n-2}{4(n-1)} S_g$ (Esposito-Pistoia -Vétois, 2013)	(M, g), $n \geq 6$, non conf. flat, $Y_g > 0$, $h_\alpha \overset{C^r}{\to} \frac{n-2}{4(n-1)} S_g$ (Robert-Vétois, 2013)	(M, g), $n \geq 6$, h_0 equals $\frac{n-2}{4(n-1)} S_g$ at order 1 at one point, $h_\alpha \overset{C^\infty}{\to} h_0$, 1-Bubble (Robert-Vétois, 2019)	(S^n, g_0), $n \geq 5$, $h_\alpha \equiv \lambda$, $\lambda > \frac{n(n-2)}{4}$ (Chen-Wei-Yan, 2011)
(S^3, g_0), $\exists (\theta_k)_k$ res. states $\theta_1 = \frac{3}{4} < \theta_k$, $k \geq 2$, $\theta_k \to +\infty$, $\lambda_\alpha \to \theta_k$ (Hebey-Wei, 2012)	(M, g), $n \geq 7$, conf. flat, $Y_g > 0$, Towering, $h_\alpha \overset{C^\infty}{\to} \frac{n-2}{4(n-1)} S_g$ (Premoselli, 2020)		(M, g), $n \geq 6$, non conf. flat, $Y_g > 0$, $h_\alpha \overset{C^r}{\to} \frac{n-2}{4(n-1)} S_g$ (Esposito-Pistoia- Vétois, Robert-Vétois, 2013)
$(S^n, \tilde{g})$, $\tilde{g}$ non conf. flat, $n \geq 25$, $h_\alpha \equiv \frac{n-2}{4(n-1)} S_{\tilde{g}}$ (Brendle, 2008; Brendle-Marques, 2009)	(M, g), non. conf. flat, $n \geq 7$, $h_\alpha \equiv \varepsilon_\alpha$ (Pistoia-Vaira, 2019)		(S^4, g_0), $h_\alpha \equiv \lambda$, $\lambda > 2$ (Vétois-Wang, 2019)
(M, g), $n \geq 4$, conf. flat, $h_\alpha \overset{C^\infty}{\to} \frac{n-2}{4(n-1)} \max_M S_{\tilde{g}}$ for some $\tilde{g} \in [g]$, $S_{\tilde{g}}$ max. at only one point (Hebey-Vaugon, 2001)	(M, g), $n = 4, 5$, $Y_g > 0$, $h_0 > \frac{n-2}{4(n-1)} S_g$ at one of the blow-up points, $h_\alpha \overset{C^\infty}{\to} h_0$ (Premoselli-Thizy, 2018)		With diagonal arguments, cf. also (Morabito-Pistoia -Vaira, 2017), (Pistoia-Vaira, 2019), (Thizy-Vétois, 2018), (Premoselli, 2020)
	(M, g), $n = 4, 5$, $h_\alpha \equiv \varepsilon_\alpha$, $S_g < 0$ at the blow-up point, (Thizy-Vétois, 2018)		
(M, g), $n \geq 4$, h_0 equals $\frac{n-2}{4(n-1)} S_g$ at order 1 at one point, $h_\alpha \overset{C^\infty}{\to} h_0$, (Robert-Vétois, 2019)	(M, g), symmetries, $n \geq 7$, non conf. flat, Towering, $h_\alpha \overset{C^\infty}{\to} \frac{n-2}{4(n-1)} S_g$ (Morabito-Pistoia-Vaira, 2017)		
(M, g), $n = 4, 5$, $S_g < 0$, $h_\alpha \equiv \varepsilon_\alpha$ (Thizy, 2016)	(M, g), $n \geq 6$, $Y_g > 0$, $h_\alpha \overset{C^\infty}{\to} \frac{n-2}{4(n-1)} S_g$ (Esposito-Pistoia-Vétois, 2013)		

Stable situations

Bounded stability	Analytic stability
(M, g), $n \geq 3$, $Y_g > 0$, $h < \frac{n-2}{4(n-1)} S_g \quad (\star)$ (Li-Zhu, $n = 3$, 1999; Druet, $n \geq 4$, 2004; see also Druet-Hebey-Vétois, 2010)	(M, g), $n \geq 4$ $\Delta_g + h$ coercive, $n \neq 6$, $h \neq \frac{n-2}{4(n-1)} S_g$ (Druet, 2003; see also Druet-Hebey, 2009)
Compactness	
(M, g) conf. flat or $3 \leq n \leq 24$, $Y_g > 0$, $(M, g) \neq (S^n, g_0)$, $h_\alpha \equiv \frac{n-2}{4(n-1)} S_g$ (Schoen, 1991; Khuri-Marques- Schoen, 2009)	

Note: when $n = 3$ the positivity of the mass of $\Delta_g + h$ can replace $(\star)$.

A general reference

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