Elliptic stability for stationary Schrödinger equations

by

Emmanuel Hebey

Part II/III

Blow-up theories.

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PART II. A priori blow-up theories in the critical case of the stationary Schrödinger equation.

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NOTE : The blue writing is what you have to write down to be able to follow the slides presentation.

PART II. A PRIORI BLOW-UP THEORIES IN THE CRITICAL CASE OF THE STATIONARY SCHRÖDINGER EQUATION.

II.1) Preliminary material :

The Sobolev inequality $\dot{H}^1 \subset L^{2^\star}$ in $\mathbb{R}^n$ is written as $\|u\|_{L^{2^\star}} \leq K_n \|\nabla u\|_{L^2}$. The precise value of the sharp constant K_n was computed by Aubin and Talenti and it was found that

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}} ,$$

where ω_n is the volume of the unit n-sphere. The extremal functions are known. They form a $(n+1)$-parameter family given by

$$u_{\Lambda,x_0}(x) = \left(\frac{\Lambda}{\Lambda^2 + \frac{|x-x_0|^2}{n(n-2)}} \right)^{\frac{n-2}{2}}$$

for $\Lambda > 0$ and $x_0 \in \mathbb{R}^n$. By an important result of Caffarelli-Gidas-Spruck (see also Obata), these extremals are the sole nonnegative solutions of the critical equation $\Delta u = u^{2^\star - 1}$ in $\mathbb{R}^n$ (which has nontrivial solutions by opposition to its subcritical version $\Delta u = u^{p-1}$, $p < 2^\star$).

II.2) The general question :

Let (M, g) closed, $n \geq 3$. Let $(h_\alpha)_\alpha$ be a converging sequence of functions (in a space to be defined according to the theory we deal with). We consider the following family of critical model equations

$$\Delta_g u + h_\alpha u = u^{2^\star - 1} , \qquad (E_\alpha)$$

and the goal we want to achieve in this second part is the following.

Goal : describe the asymptotic behaviour of sequences $(u_\alpha)_\alpha$ of solutions of the (E_α)'s in reasonnable spaces.

There will be three "reasonnable spaces" : $L^{2^\star}$, H^1, and C^0 (or C^k, $k \geq 1$) leading to the

- L^p-theory ($L^{2^\star}$-description of the asymptotics),
- H^1-theory (H^1-description of the asymptotics),
- C^0-theory (pointiwse estimates).

The two first theories are concerned with slightly more general objects (Palais-Smale sequences) than sequences of solutions (there is room in $L^{2^\star}$ and in H^1 to add small H^1-terms leading to the notion of Palais-Smale sequences).

Solutions of (E_α) can be seen as (nonnegative) critical points of the free functionals $I_\alpha : H^1 \to \mathbb{R}$ given by

$$I_\alpha(u) = \frac{1}{2}\int_M \left(|\nabla u|^2 + h_\alpha u^2\right) dv_g - \frac{1}{2^\star}\int_M |u|^{2^\star} dv_g \ .$$

A sequence $(u_\alpha)_\alpha$ in H^1 is said to be a Palais-Smale sequence (for short PS sequence) for (E_α) if :

(i) $\left((I_\alpha(u_\alpha)\right)_\alpha$ is bounded in $\mathbb{R}$,
(ii) $DI_\alpha(u_\alpha) \to 0$ in $(H^1)^\star$ as $\alpha \to +\infty$.

The two equations $I_\alpha(u_\alpha) = O(1)$, $DI_\alpha(u_\alpha).(u_\alpha) = o\left(\|u_\alpha\|_{H^1}\right)$ imply that Palais-Smale sequences are bounded in H^1 (Brézis-Nirenberg). Conversely, a sequence of solutions of (E_α), in the sense that each u_α solves (E_α), which is bounded in H^1, is a Palais-Smale sequence. The Palais-Smale sequence notion extends the notion of H^1-bounded sequences of solutions by relaxing the condition $DI_\alpha(u_\alpha) = 0$ into $DI_\alpha(u_\alpha) \to 0$.

II.3) The L^p-theory :

The L^p-theory describes the asymptotical behaviour (the blow-up) in $L^{2^\star}$ in terms of Dirac masses. The theory goes back to P.L.Lions and can be seen as an easy consequence of his concentration-compactness principle.

Theorem : (Concentration-Compactness, P.L.Lions, 84)

Let $(u_\alpha)_\alpha$ be a bounded sequence in H^1 such that $u_\alpha \rightharpoonup u_\infty$ in H^1, and such that the measures $\mu_\alpha = |\nabla u_\alpha|^2 dv_g \rightharpoonup \mu$ and $\nu_\alpha = |u_\alpha|^{2^\star} dv_g \rightharpoonup \nu$ converge weakly in the sense of measures. Then there exist an at most countable set J, distinct points $x_j \in M$ for $j \in J$, and positive real numbers $\mu_j, \nu_j > 0$ for $j \in J$ such that

$$\nu = |u_\infty|^{2^\star} dv_g + \sum_{j \in J} \nu_j \delta_{x_j} \ ,$$

$$\mu \geq |\nabla u_\infty|^2 dv_g + \sum_{j \in J} \mu_j \delta_{x_j} \ ,$$

and such that $\frac{1}{K_n^2} \nu_j^{2/2^\star} \leq \mu_j$, where K_n is the sharp constant in the Sobolev inequality. In particular, $\sum_{j \in J} \nu_j^{2/2^\star} < +\infty$.

We assume here that the h_α's converge in L^∞. Let h_∞ be the limit of the h_α's so that $h_\alpha \to h_\infty$ in L^∞ as $\alpha \to +\infty$. Then the (E_α)'s have, as a formal limit equation, our model equation

$$\Delta_g u + h_\infty u = u^{2^\star - 1} . \qquad (E_\infty)$$

The L^p-theory theorem is stated as follows.

Theorem : (L^p-theory, P.L.Lions, 84)

Let $(h_\alpha)_\alpha$ be a sequence in L^∞ such that $h_\alpha \to h_\infty$ as $\alpha \to +\infty$, and $(u_\alpha)_\alpha$ a PS sequence of nonnegative functions for (E_α). There exists $u_\infty \in H^1$, a nonnegative solution of (E_∞), $N \in \mathbb{N}$, $x_1, \ldots, x_N \in M$, and $\lambda_1, \ldots, \lambda_N > 0$ such that, up to a subsequence,

$$u_\alpha^{2^\star} dv_g \rightharpoonup u_\infty^{2^\star} dv_g + \sum_{i=1}^{N} \lambda_i \delta_{x_i} \qquad (L^p E)$$

weakly in the sense of measures. Moreover, $u_\alpha \rightharpoonup u_\infty$ in H^1.

The x_i's are referred to as the geometric blow-up points of the sequence $(u_\alpha)_\alpha$. As a direct consequence of the theorem, since $\int u_\alpha^{2^\star} \to \int u_\infty^{2^\star}$ outside the x_i's, we get that (Brézis-Lieb) $u_\alpha \to u_\infty$ in $H^1_{loc}(M \backslash \mathcal{S})$, where $\mathcal{S} = \{x_i, i = 1, \ldots, N\}$.

Proof of the theorem : Since $(u_\alpha)_\alpha$ is bounded in H^1 we can assume that, up to a subsequence, $u_\alpha \rightharpoonup u_\infty$ in H^1, $u_\alpha \to u_\infty$ in L^2, and $u_\alpha \to u_\infty$ a.e. It is easily seen (Yamabe) that u_∞ solves (E_∞). Let $\mu_\alpha = |\nabla u_\alpha|^2 dv_g$ and $\nu_\alpha = |u_\alpha|^{2^\star} dv_g$. By the weak compactness of measures, $\mu_\alpha \rightharpoonup \mu$ and $\nu_\alpha \rightharpoonup \nu$ as $\alpha \to +\infty$. Let $\varphi \in C^\infty$. By the PS property, $DI_\alpha(u_\alpha).(\varphi u_\alpha) = o(1)$, and thus

$$\int (\nabla u_\alpha \nabla(\varphi u_\alpha)) + \int h_\alpha \varphi u_\alpha^2 = \int |u_\alpha|^{2^\star} \varphi + o(1) \, . \tag{1}$$

We compute

$$\int (\nabla u_\alpha \nabla(\varphi u_\alpha)) = \mu(\varphi) + \int u_\infty(\Delta_g u_\infty)\varphi - \int |\nabla u_\infty|^2 \varphi + o(1) \, ,$$

and $\int h_\alpha \varphi u_\alpha^2 = \int h_\infty \varphi u_\infty^2 + o(1)$. There holds $\int |u_\alpha|^{2^\star}\varphi = \nu(\varphi) + o(1)$. By CCP $\mu \geq |\nabla u_\infty|^2 dv_g + \sum_{j\in J} \mu_j \delta_{x_j}$ and $\nu = |u_\infty|^{2^\star} dv_g + \sum_{j\in J} \nu_j \delta_{x_j}$. Since u_∞ solve (E_∞), we get from (1) that

$$\sum_{j\in J} \mu_j \varphi(x_j) \leq \sum_{j \in J} \nu_j \varphi(x_j) \tag{2}$$

for all $\varphi \in C^\infty$. The series $\sum \mu_j$ and $\sum \nu_j$ converge and thus $\mu_j \leq \nu_j$ for all j. By CCP, $\frac{1}{K_n^2}\nu_j^{2/2^\star} \leq \mu_j$. Thus $\nu_j^{1-\frac{2}{2^\star}} \geq K_n^{-2}$ and J has to be finite. This proves (L^pE). Q.E.D.

II.3) The H^1-theory :

The question now is to understand what kind of objects are hidden behind the Dirac masses of the L^p-theory, namely to understand the α-dynamics of formation of the Dirac masses in the L^p-theory. The key notion there is that of a bubble (or sphere singularity). The definition is as follows.

Definition : (Bubble)

A bubble is a sequence $(B_\alpha)_\alpha$ of functions, $B_\alpha : M \to \mathbb{R}$, given by

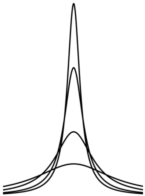
$$B_\alpha(x) = \left(\frac{\mu_\alpha}{\mu_\alpha^2 + \frac{d_g(x_\alpha, x)^2}{n(n-2)}} \right)^{\frac{n-2}{2}}$$

for all $x \in M$ and all α, where d_g is the Riemannian distance, $(x_\alpha)_\alpha$ is a converging sequence of points in M, and $(\mu_\alpha)_\alpha$, $\mu_\alpha \to 0$, is a sequence of positive real numbers converging to zero as $\alpha \to +\infty$.

The x_α's are the centers of the bubble. The μ_α's are the weights of the bubble.

Up to changing the Riemannian distance d_g by the Euclidean distance in the definition of a bubble, we recognize in this definition the extremal functions u_{Λ,x_0} for the sharp Euclidean Sobolev inequality. These, as already mentioned, are also (Caffarelli-Gidas-Spruck) the sole nonnegative solutions of $\Delta u = u^{2^\star-1}$ in $\mathbb{R}^n$ (which is the limit equation we get by blowing-up the (E_α)'s as in the Gidas-Spruck "baby" blow-up argument).

Let $x_0 = \lim_\alpha x_\alpha$. Then $B_\alpha \to 0$ in $L^\infty_{loc}(M\backslash\{x_0\})$. We can compute that, actually, $B_\alpha \to 0$ in L^∞ in $M\backslash B_{x_\alpha}(r_\alpha)$ when $r_\alpha \gg \sqrt{\mu_\alpha}$. On the other hand, since $B_\alpha(x_\alpha) = \frac{1}{\mu_\alpha^{(n-2)/2}}$, we get that $B_\alpha(x_\alpha) \to +\infty$ as $\alpha \to +\infty$.



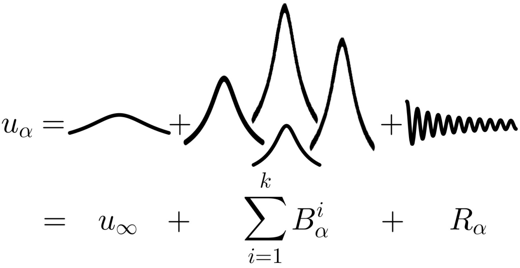
There holds $B_\alpha^{2^\star} dv_g \rightharpoonup \frac{1}{K_n^n}\delta_{x_0}$, where K_n is the sharp constant in the Euclidean Sobolev inequality. Bubbles are perfect candidates to be hidden behind the Dirac masses of the L^p-theory.

Theorem : (H^1-theory, M. Struwe, 84)

Let $(h_\alpha)_\alpha$ be a sequence in L^∞ such that $h_\alpha \to h_\infty$ as $\alpha \to +\infty$, and $(u_\alpha)_\alpha$ be a PS sequence of nonnegative functions for (E_α). There exist $u_\infty \in H^1$, a nonnegative solution of (E_∞), $k \in \mathbb{N}$, and k bubbles $(B^i_\alpha)_\alpha$, $i = 1, \ldots, k$, such that, up to a subsequence,

$$u_\alpha = u_\infty + \sum_{i=1}^{k} B^i_\alpha + R_\alpha \ , \qquad (H^1E)$$

where $(R_\alpha)_\alpha$ is a sequence in H^1 such that $R_\alpha \to 0$ in H^1 as $\alpha \to +\infty$.



The bubbles in (H^1E) do not interact one with another (at the H^1-level) and (H^1E) comes with an important equation (that we refer to as the structure equation) which implies that the H^1-scalar product between two bubbles $(B^i_\alpha)_\alpha$ and $(B^j_\alpha)_\alpha$ in (H^1E) tends to zero as $\alpha \to +\infty$. The structure equation is written as

$$\frac{\mu_{i,\alpha}}{\mu_{j,\alpha}} + \frac{\mu_{j,\alpha}}{\mu_{i,\alpha}} + \frac{d_g(x_{i,\alpha}, x_{j,\alpha})^2}{\mu_{i,\alpha}\mu_{j,\alpha}} \to +\infty \qquad (SE)$$

as $\alpha \to +\infty$, for all $i \neq j$, where the $x_{i,\alpha}$'s and $\mu_{i,\alpha}$'s are the centers and weights of the bubbles $(B^i_\alpha)_\alpha$. In particular, the $\|\cdot\|^2_{H^1}$ and $\|\cdot\|^{2^\star}_{L^{2^\star}}$ norms of the u_α's respect the decomposition (H^1E).

We can check $(H^1E) \Rightarrow (L^pE)$, where N in (L^pE) is the number of distinct limits we get by the convergence of the centers $x_{i,\alpha}$ of the bubbles (there may be bubbles which accumulate one on another, and thus that $k > N$), where the x_i's, $i = 1, \ldots, N$, in (L^pE) are the limits of the $x_{j,\alpha}$'s, $j = 1, \ldots, k$, where the λ_i's in (L^pE) are given by $\lambda_i = n_i K_n^{-n}$ for all $i = 1, \ldots, N$, and where n_i is the number of $x_{j,\alpha}$'s, $j = 1, \ldots, k$, which converge to x_i so that $\sum_{i=1}^N n_i = k$ (grape decomposition).

Brief sketch of proof of the theorem : There are two preliminary lemmas (easy), one inductive lemma (difficult), and one concluding lemma (easy). The two preliminary lemmas are as follows :

(L1) (Brézis-Nirenberg, Yamabe) PS sequences $(u_\alpha)_\alpha$ are bounded in H^1 and, up to a subsequence, they converge weakly in H^1, strongly in L^2, and a.e. to some u_∞ which solves (E_∞).

and

(L2) If $(u_\alpha)_\alpha$ is a PS sequence for (E_α), and $u_\alpha \rightharpoonup u_\infty$ in H^1, then $v_\alpha = u_\alpha - u_\infty$ is a PS sequence for the free functional

$$I_0(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{2^\star}|u|^{2^\star} \ ,$$

there holds that $v_\alpha \rightharpoonup 0$ in H^1, and

$$I_0(v_\alpha) = I_\alpha(u_\alpha) - I_\infty(u_\infty) + o(1)$$

for all α, where $I_\infty = \lim I_\alpha$.

Roughly speaking (L2) states that we can get rid of the limit profile u_∞ and the potentials h_α which, both, can be set to zero.

The key inductive lemma is as follows.

(L_{ind}) Let $(v_\alpha)_\alpha$ be a PS sequence of nonnegative functions for I_0 such that $v_\alpha \rightharpoonup 0$ in H^1 but $v_\alpha \not\to 0$ in H^1 as $\alpha \to +\infty$. There exist a bubble $(B_\alpha)_\alpha$, and a PS sequence $(w_\alpha)_\alpha$ of nonnegative functions for I_0 such that, up to a subsequence,

$$w_\alpha = v_\alpha - B_\alpha + R_\alpha$$

for all α, where $R_\alpha \to 0$ in H^1 as $\alpha \to +\infty$, and

$$I_0(w_\alpha) = I_0(v_\alpha) - \frac{1}{nK_n^n} + o(1)$$

for all α (the constant $\frac{1}{nK_n^n}$ being precisely, up to $o(1)$, the free energy $I_0(B_\alpha)$ of the bubble).

At last the concluding lemma is as follows.

(L3) (Aubin, Brézis-Nirenberg) Let $(v_\alpha)_\alpha$ be a PS sequence of nonnegative functions for I_0 such that $v_\alpha \rightharpoonup 0$ in H^1 and $I_0(v_\alpha) \to c$ as $\alpha \to +\infty$. If $c < \frac{1}{nK_n^n}$, then $v_\alpha \to 0$ in H^1 as $\alpha \to +\infty$.

With these lemmas (L1), (L2), (L3), and (L_{ind}) we can prove the theorem.

Brief sketch of proof of the theorem (continued) : Let $(u_\alpha)_\alpha$ be a PS sequence of nonnegative functions for (E_α). By (L1) and (L2), the u_α's are bounded in H^1, $u_\alpha \rightharpoonup u_\infty$ in H^1, $u_\alpha \to u_\infty$ in L^2, $u_\alpha \to u_\infty$ a.e., where $u_\infty \geq 0$ solves (E_∞). Moreover, $v_\alpha = u_\alpha - u_\infty$ is a PS sequence for I_0 such that $v_\alpha \rightharpoonup 0$ in H^1 and

$$I_0(v_\alpha) = I_\alpha(u_\alpha) - I_\infty(u_\infty) + o(1) \ .$$

A nice (tricky though easy) argument shows that we can assume that $v_\alpha \geq 0$ for all α (up to adding a $R_\alpha \to 0$ in H^1 to the v_α's). We let $w_\alpha^0 = v_\alpha$. In case $w_\alpha^0 \to 0$ in H^1, we have the theorem with $k = 0$. If not the case $w_\alpha^0 \not\to 0$ in H^1 and by (L_{ind}) there exist a bubble $(B_\alpha^1)_\alpha$, and a PS sequence $(w_\alpha^1)_\alpha$ of nonnegative functions for I_0 such that, up to a subsequence,

$$\begin{aligned} w_\alpha^1 &= w_\alpha^0 - B_\alpha^1 + R_\alpha \ , \text{ and} \\ I_0(w_\alpha^1) &= I_0(w_\alpha^0) - \frac{1}{nK_n^n} + o(1) \end{aligned}$$

for all α, where $R_\alpha \to 0$ in H^1. Clearly, $w_\alpha^1 \rightharpoonup 0$ in H^1. Here again, either $w_\alpha^1 \to 0$ in H^1, and we get the theorem with $k = 1$, or $w_\alpha^1 \not\to 0$ in H^1 and we can apply again (L_{ind}). We go on with this procedure.

At the k^{th} step we get k bubbles $(B^i_\alpha)_\alpha$, and a PS sequence $(w^k_\alpha)_\alpha$ of nonnegative functions for I_0 such that, up to a subsequence,

$$w^k_\alpha = w^0_\alpha - \sum_{i=1}^{k} B^i_\alpha + R_\alpha \ ,$$

$$I_0(w^k_\alpha) = I_0(w^0_\alpha) - \frac{k}{nK_n^n} + o(1)$$

for all α, where $R_\alpha \to 0$ in H^1. By (L3), $w^k_\alpha \to 0$ in H^1 if $I_0(w^k_\alpha) \to c$ with $c < \frac{1}{nK_n^n}$. Obviously this implies that the process has to step at some stage since, at each step, we substract a fix amount of energy $(1/nK_n^n)$ to the initial energy $I_\alpha(u_\alpha) - I_\infty(u_\infty)$. When the process stops,

$$w^0_\alpha - \sum_{i=1}^{k} B^i_\alpha = R_\alpha$$

and this is precisely (H^1E) since $w^0_\alpha = u_\alpha - u_\infty$. The theorem is proved. Q.E.D.

II.4) The C^0-theory :

At this stage we would like to get sharper estimates involving pointwise asymptotics. For this we need to drop dealing with PS sequences (since PS sequences are stable by the addition of R_α's when $R_\alpha \to 0$ in H^1). The C^0-theory has to do with H^1-bounded sequences of solutions of (E_α). Namely with sequences $(u_\alpha)_\alpha$ of nonnegative functions such that

$$\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{2^\star - 1} \qquad (E_\alpha)$$

and $\|u_\alpha\|_{H^1} = O(1)$ for all α. Two remarks are in order.

Rk1 : (In general, $R_\alpha \not\to 0$ in L^∞). The naive idea stating that the C^0-theory is just the H^1-theory with the rest $(R_\alpha)_\alpha$ converging to zero in L^∞ is false. The solutions u_α of the Yamabe equation on the sphere, which can be made to blow up, have an H^1-decomposition with one bubble like $u_\alpha = B_\alpha + R_\alpha$, $R_\alpha \to 0$ in H^1, but (as we can compute) $R_\alpha \not\to 0$ in L^∞ when $n \geq 6$, while we even have that $\|R_\alpha\|_{L^\infty} \to +\infty$ when $n \geq 7$. In other words, we will have to come with something which is slightly more subtle than the sole convergence of the rest to zero in L^∞.

The second remark is even more important.

Rk2 : (Bubbles in the H^1-decomposition may interact at the C^0-level). As already mentionned, because of the structure equation (SE), bubbles do not interact one with another at the H^1-level. The point here is that a bubble $(B_\alpha)_\alpha$, with centers and weights x_α and μ_α, live in the H^1-world essentially in the balls $B_{x_\alpha}(r_\alpha)$ for $r_\alpha \approx \mu_\alpha$. In particular, it H^1-dies outside such balls :

$$\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\int_{M\backslash B_{x_\alpha}(R\mu_\alpha)}|\nabla B_\alpha|^2 dv_g = 0 \ .$$

On the other hand, the B_α's live up to $\sqrt{\mu_\alpha}$ in the C^0-world : for any $R > 0$, there exists $\varepsilon_R > 0$ such that

$$\inf_{B_{x_\alpha}(R\sqrt{\mu_\alpha})} B_\alpha \geq \varepsilon_R$$

for all α. Since $\sqrt{\mu_\alpha} \gg \mu_\alpha$ for $\alpha \gg 1$, there is a whole region in which we see bubbles in C^0, and where bubbles may interact one with another even though they do not interact at the H^1-level. Any (a priori) C^0-theory will have to take care of the possible interactions of bubbles at the C^0-level.

Theorem : (C^0-theory, Druet-H.-Robert, 2004)

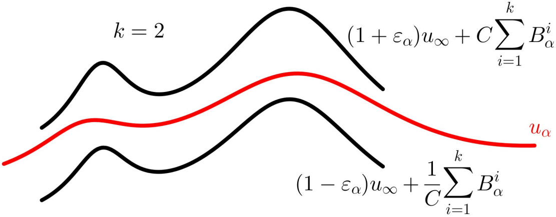
Let $(h_\alpha)_\alpha$ be a sequence in $C^{0,\theta}$ converging in $C^{0,\theta}$ to some h_∞. Let $(u_\alpha)_\alpha$ be a bounded sequence in H^1 of solutions of (E_α). Assume that $\Delta_g + h_\infty$ is coercive. There exist $k \in \mathbb{N}$, a nonnegative solution u_∞ of the limit equation (E_∞), and k bubbles $(B^i_\alpha)_\alpha$, $i = 1, \ldots, k$, such that, up to a subsequence,

$$\begin{aligned} &(1-\varepsilon_\alpha)u_\infty(x) + \frac{1}{C}\sum_{i=1}^k B^i_\alpha(x) \\ &\leq u_\alpha(x) \leq (1+\varepsilon_\alpha)u_\infty(x) + C\sum_{i=1}^k B^i_\alpha(x) \end{aligned} \tag{C^0E}$$

for all $x \in M$ and all α, where $C > 1$ is independent of α and x, and (ε_α) is a sequence of positive real numbers, independent of x, such that $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$. Moreover, (H^1E) holds true with u_∞, k, and these $(B^i_\alpha)_\alpha$.

Rk : The condition that $\Delta_g + h_\infty$ should be coercive is a necessary (and sufficient) condition in order to get (C^0E).

In other words (e.g. when $k = 2$) :



Moreover

$$u_\alpha = u_\infty + \sum_{i=1}^{k} B_\alpha^i + R_\alpha \ ,$$

where $R_\alpha \to 0$ in H^1. In some sense, though (see Rk1 above) we cannot formally assume that $\|R_\alpha\|_{L^\infty} \to 0$ in (H^1E), the C^0-theory provides sharp upper and lower bounds where $R_\alpha \equiv 0$.

The theorem has another variant we get from the Green's representation of the u_α's which gives the exact asymptotic formula for the u_α's.

Let G_∞ be the Green's fct of $\Delta_g + h_\infty$ and $\Phi : M \times M \to \mathbb{R}^+$ be given by

$$\Phi(x,y) = (n-2)\omega_{n-1} d_g(x,y)^{n-2} G_\infty(x,y) ,$$

where ω_{n-1} is the volume of the $(n-1)$-sphere. Then Φ is continuous in $M \times M$ and $\Phi = 1$ on the diagonal.

Theorem : (C^0 exact asymptotic formula, Druet-H.-Robert, 2004)

For any sequence $(x_\alpha)_\alpha$ in M,

$$u_\alpha(x_\alpha) = \Big(1 + o(1)\Big) u_\infty(x_\alpha) + \sum_{i=1}^{k} \Big(\Phi(x_i, x_\alpha) + o(1)\Big) B_\alpha^i(x_\alpha) ,$$

where the x_i's are the limits of the centers of the B_α^i's.

In particular, the constant C in (C^0E) can be taken as close as we want to 1 when standing in small balls $B_{x_i}(\delta)$, $0 < \delta \ll 1$.

Of course, C^0-theory $\Rightarrow$ H^1-theory $\Rightarrow$ L^p-theory (when we restrict ourselves to sequences of solutions and not only to PS sequences).

Thank you for your attention !