A SHORT (INFORMAL) INTRODUCTION TO BOPP-PODOLSKY-SCHRÖDINGER-PROCA AND SCHRÖDINGER-POISSON-PROCA SYSTEMS IN THE ELECTRO-MAGNETO-STATIC CASE

EMMANUEL HEBEY

Abstract. We discuss Bopp-Podolsky-Schrödinger-Proca and Schrödinger-Poisson-Proca systems in the case of electro-magneto-static solutions when the background space is a closed 3-manifold. We present recent results we obtained on these systems.

The Bopp-Podolsky theory, developed by Bopp [10], and independently by Podolsky [29], is a second order gauge theory for the electromagnetic field which refines the Maxwell theory. When coupled with the Schrödinger equation it aims (as for the Maxwell-Schrödinger theory) to describe the evolution of a charged nonrelativistic quantum mechanical particle interacting with the electromagnetic field it generates. In this theory the electromagnetic field is both generated by and drives the particle field.

We are going to discuss two systems in this survey. One is the Bopp-Podolsky-Schrödinger-Proca reduced system $(BPSP)_a$ given by

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \Phi(x,v,A)u = u^{p-1} \\ a^2\Delta_g^2 v + \Delta_g v + m_1^2 v = 4\pi q u^2 \\ a^2\Delta_g^2 A + \Delta_g A + m_1^2 A = \frac{4\pi q\hbar}{m_0^2}\Psi(A,S)u^2 \end{cases} \qquad (BPSP)_a$$

with unknowns (u, v, A), where u and v are functions, $u \geq 0$ in M, and A is a 1-form. Basically (see Section 1), u corresponds to the amplitude of the particle field that we write in polar form, (v, A) represent the electromagnetic field that the particle field creates and the whole system corresponds to an electro-magneto-static regime. In the above equations,

$$\Phi(x,v,A) = \frac{\hbar^2}{2m_0^2}|\Psi(A,S)|^2 + \omega^2 + qv \ ,$$
$$\Psi(A,S) = \nabla S - \frac{q}{\hbar}A \ ,$$

$a, q, m_0, m_1 > 0$ are positive real numbers and $\omega \in \mathbb{R}$. Also $\Delta_g = -\mathrm{div}_g\nabla$ is the Laplace-Beltrami operator when acting on functions u and v, $\Delta_g = \delta d + d\delta$ is the Hodge-de Rham Laplacian when acting on 1-forms A, $\hbar$ is the reduced Planck's constant and $p \in (2, 6]$. Following standard notations, d is the differential, $\delta = -\nabla.$ is the codifferential (it depends on g, we could have written δ_g to be coherent with the Δ_g notation, but this is not a very common notation) and 6 is the critical

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Sobolev exponent (when $n = 3$). The Coulomb gauge equation is $\delta A = 0$. We let $(\overline{BPSP})_a$ be the saturated reduced system of four equations given by

$$(\overline{BPSP})_a = (BPSP)_a + \text{“}\delta A = 0\text{”} \ .$$

The positive real number a in these equations is the Bopp-Podolsky parameter. Systems of equations like $(BPSP)_a$ and $(\overline{BPSP})_a$ are derived from a larger Bopp-Podolsky-Schrödinger-Proca system as we will see in Section 1. The second system corresponds to $a = 0$. We then get what is referred to as the Schrödinger-Poisson-Proca reduced system (SPP) given by

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \Phi(x,v,A)u = u^{p-1} \\ \Delta_g v + m_1^2 v = 4\pi q u^2 \\ \Delta_g A + m_1^2 A = \frac{4\pi q\hbar}{m_0^2}\Psi(A,S)u^2 \ . \end{cases} \qquad (SPP)$$

Here again the unkowns are (u, v, A) and we require that $u \geq 0$. As before we let $(\overline{SPP})$ be the saturated reduced system of four equations given by

$$(\overline{SPP}) = (SPP) + \text{“}\delta A = 0\text{”}$$

that we get from (SPP) by adding to it the Coulomb gauge condition $\delta A = 0$. As above systems of equations like (SPP) and $(\overline{SPP})$ are derived from a larger system (see Section 1). As a remark we chose to study our systems in the context of closed 3-manifolds. This will have some impact as we will see below.

1. Construction of the equations

We use Lagrangian constructions. The particle field is here represented by a function ψ and the electromagnetic field is represented by a gauge potential (A, φ), where φ (a function) represents the electric field and A (a 1-form) represents the magnetic field that the particle field creates. We adopt here the m_1-Proca formalism meaning that a mass is given to the electromagnetic field (φ, A). The particle field ψ is ruled by a nonlinear Schrödinger equation. The electromagnetic field (φ, A) is ruled by the Bopp-Podolsky-Proca action in the Bopp-Podolsky-Proca model. We need then to couple the Schrödinger and the Bopp-Podolsky-Proca actions. This is done by using the minimum coupling rule

$$\partial_t \to \tilde{\partial}_t = \partial_t + i\frac{q}{\hbar}\varphi \ \ , \ \ \nabla \to \tilde{\nabla} = \nabla - i\frac{q}{\hbar}A \ , \tag{1.1}$$

where q and m_0 are the charge and mass of ψ. The minimum coupling rule is the rule traditionnally used in electrodynamics to account for all electromagnetic interactions. The nonlinear Schrödinger Lagrangian for ψ is then given by

$$\begin{aligned} \mathcal{L}_{NLS} &= i\hbar\frac{\partial\psi}{\partial t}\overline{\psi} - q\varphi|\psi|^2 - \frac{\hbar^2}{2m_0^2}|\nabla\psi - i\frac{q}{\hbar}A\psi|^2 + \frac{2}{p}|\psi|^p \\ &= i\hbar\frac{\tilde{\partial}\psi}{\partial t}\overline{\psi} - \frac{\hbar^2}{2m_0^2}|\tilde{\nabla}\psi|^2 + \frac{2}{p}|\psi|^p \ . \end{aligned} \tag{1.2}$$

This is nothing but the usual nonlinear Schrödinger Lagrangian when time and space derivative are given by the coupling (1.1). It remains now to write down the Bopp-Podolsky-Proca Lagrangian for the field (φ, A). We assume in this section (and only in this section and in Sections 3 and 5) that our manifold is orientable. We then define the Bopp-Podolsky-Proca Lagrangian $\mathcal{L}_{BPP}$ by

$$\begin{aligned}\mathcal{L}_{BPP}(\varphi, A) = \frac{1}{8\pi}\left|\frac{\partial A}{\partial t} + \nabla\varphi\right|^2 - \frac{1}{8\pi}|\nabla \times A|^2 \\ + \frac{m_1^2}{8\pi}\left(|\varphi|^2 - |A|^2\right) + \frac{a^2}{8\pi}\mathcal{L}_{Add}(\varphi, A)\ ,\end{aligned} \tag{1.3}$$

where $a \in \mathbb{R}^+$, $\nabla\times = \star d$ is the curl operator ($\star$ is the Hodge dual and d the usual differentiation on forms),

$$\mathcal{L}_{Add}(\varphi, A) = \left(-\Delta_g\varphi + \nabla.\partial_t A\right)^2 - \left|\overline{\Delta}_g A + \partial_t\left(\nabla\varphi + \partial_t A\right)\right|^2$$

and $\overline{\Delta}_g = \nabla \times \nabla\times = \delta d$ is half the Hodge-de Rham Laplacian for 1-forms (δ is the codifferential). The blue part in (1.3) is the Maxwell part. The red part in (1.3) is the Proca part. The orange part in (1.3) is the Bopp-Podolsky part. As already mentioned a is the Bopp-Podolsky parameter (physically to be small). Both Proca and Bopp-Podolsky are then corrections of the Maxwell theory. As a remark,

$$\|(\varphi, A)\|^2_{\text{Lorentz}} = |\varphi|^2 - |A|^2\ ,$$

where the LHS is the Lorentz norm. Therefore we are indeed giving a mass m_1 to the field (φ, A) in the red part (the Proca part) of (1.3). Once we have $\mathcal{L}_{NLS}$ and $\mathcal{L}_{BPP}$ we define the total action functional $\mathcal{S}_{tot}$ by

$$\mathcal{S}_{tot} = \int\int\left(\mathcal{L}_{NLS} + \mathcal{L}_{BPP}\right)dv_g dt\ .$$

Assuming that ψ is of the form $\psi = ue^{iS}$ (polar form) with $u \geq 0$, and taking the variation of $\mathcal{S}_{tot}$ with respect to u, S, φ, and A, we get four equations which, pulled together, form the full Bopp-Podolsky-Schrödinger-Proca system

$$\begin{cases}\frac{\hbar^2}{2m_0^2}\Delta_g u + \left(\hbar\frac{\partial S}{\partial t} + q\varphi + \frac{\hbar^2}{2m_0^2}|\Psi(A,S)|^2\right)u = u^{p-1} \\ 2u\frac{\partial u}{\partial t} + \frac{\hbar}{m_0^2}\nabla.\left(\Psi(A,S)u^2\right) = 0 \\ -\frac{1}{4\pi}\nabla.\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) - \frac{a^2}{4\pi}\Delta_g M(\varphi,A) - \frac{a^2}{4\pi}\frac{\partial}{\partial t}\nabla.N(\varphi,A) + \frac{m_1^2}{4\pi}\varphi = qu^2 \\ \frac{1}{4\pi}\overline{\Delta}_g A + \frac{1}{4\pi}\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + \frac{m_1^2}{4\pi}A + \frac{a^2}{4\pi}Q(\varphi,A) = \frac{\hbar q}{m_0^2}\Psi(A,S)u^2\ ,\end{cases} \tag{1.4}$$

where

$$\begin{aligned}&\Psi(A,S) = \nabla S - \frac{q}{\hbar}A\ ,\ M(\varphi,A) = -\Delta_g\varphi + \nabla.\partial_t A\ , \\ &N(\varphi,A) = \overline{\Delta}_g A + \partial_t\left(\nabla\varphi + \partial_t A\right)\ , \\ &Q(\varphi,A) = \overline{\Delta}_g N(\varphi,A) + \frac{\partial^2}{\partial t^2}N(\varphi,A) - \nabla\frac{\partial}{\partial t}M(\varphi,A)\ .\end{aligned}$$

Letting $a = 0$ in (1.4) we get the Maxwell-Schrödinger-Proca system

$$\begin{cases}\frac{\hbar^2}{2m_0^2}\Delta_g u + \left(\hbar\frac{\partial S}{\partial t} + q\varphi + \frac{\hbar^2}{2m_0^2}|\Psi(A,S)|^2\right)u = u^{p-1} \\ 2u\frac{\partial u}{\partial t} + \frac{\hbar}{m_0^2}\nabla.\left(\Psi(A,S)u^2\right) = 0 \\ -\frac{1}{4\pi}\nabla.\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + \frac{m_1^2}{4\pi}\varphi = qu^2 \\ \frac{1}{4\pi}\overline{\Delta}_g A + \frac{1}{4\pi}\frac{\partial}{\partial t}\left(\frac{\partial A}{\partial t} + \nabla\varphi\right) + \frac{m_1^2}{4\pi}A = \frac{\hbar q}{m_0^2}\Psi(A,S)u^2\ .\end{cases} \tag{1.5}$$

Letting $m_1 = 0$ the Proca contribution disappears. In other words:

Value of a	Value of m_1	The physics behind
$a \neq 0$	$m_1 \neq 0$	Bopp-Podolsky-Schrödinger-Proca
$a = 0$	$m_1 \neq 0$	Maxwell-Schrödinger-Proca
$a \neq 0$	$m_1 = 0$	Bopp-Podolsky-Schrödinger
$a = 0$	$m_1 = 0$	Maxwell-Schrödinger

The Maxwell-Proca model is often referred to as the De Broglie-Proca model.

2. The reduced equations

We assume here that we are in the static case of the system, and therefore that $\partial_t u \equiv 0$, $\partial_t \varphi \equiv 0$ and $\partial_t A \equiv 0$. Then we look for solutions with

$$S(x,t) = S(x) + \frac{\omega^2}{\hbar} t \ .$$

Such type of solutions are referred to as electro-magneto-static solutions. We get standing waves solutions when $S(x) \equiv 0$. The above form of $S(x,t)$ was introduced in the paper [8] by Benci and Fortunato for the Klein-Gordon-Maxwell equations in $\mathbb{R}^3$ (see also d'Avenia, Mederski and Pomponio [4]). Looking for electro-magneto-static solutions, (1.4) can be written in the following form

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u + \Phi(x,\varphi,A)u = u^{p-1} \\ \nabla.\left(\Psi(A,S)u^2\right) = 0 \\ a^2\Delta_g^2\varphi + \Delta_g\varphi + m_1^2\varphi = 4\pi q u^2 \\ a^2\overline{\Delta}_g^2 A + \overline{\Delta}_g A + m_1^2 A = \frac{4\pi\hbar q}{m_0^2}\Psi(A,S)u^2 \ . \end{cases} \tag{2.1}$$

By the fourth equation in (2.1), since $\nabla.\overline{\Delta}_g = 0$ (as $\delta^2 = 0$), the second equation in (2.1) is nothing but the Coulomb gauge condition $\delta A = 0$, and the three other equations give rise to $(BPSP)_a$ by noting that when $\delta A = 0$ we get that $\overline{\Delta}_g A = \Delta_g A$, where $\Delta_g = d\delta + \delta d$ is the Hodge-de Rham Laplacian on forms. System (2.1) is the electro-magneto-static version of (1.4) and is nothing but $(\overline{BPSP})_a$.

Similarly we get (SPP) from (1.5) in the sense that (1.5) with $\partial_t u \equiv 0$, $\partial_t \varphi \equiv 0$, $\partial_t A \equiv 0$ and $S(x,t) = S(x) + \frac{\omega^2}{\hbar}t$ is nothing but $(\overline{SPP})$. We refer to $(BPSP)_a$ and $(\overline{BPSP})_a$ as the Bopp-Podolsky-Schrödinger-Proca system in the electro-magneto-static case, and to (SPP) and $(\overline{SPP})$ as the Schrödinger-Poisson-Proca system in the electro-magneto-static case.

3. The derivations in (1.4)

The derivations to get (1.4) involve elementary mathematics. The terms are quadratic, and therefore easy to derive. Basic tools complete what we need to get (1.4). We briefly discuss the derivation of the term

$$A \to \int |\nabla \times A|^2$$

from which we get the half Laplacian $\overline{\Delta}_g A$ in the equations. We get the result using basic differential calculus together with elementary Hodge de Rham theory. If we let ω_g be the volume form of (M,g), then

$$\begin{aligned}
\frac{1}{2}\left(\frac{d}{dA}\int|\nabla\times A|^2\right).(B) &= \int(\star dA,\star dB)\,\omega_g \qquad (\text{quadratic} + \nabla\times = \star d)\\
&= (-1)^{n-1}\int(\star dA,(\star d\star)\star B)\,\omega_g \qquad (\star\star = (-1)^{n-1} \text{ in } \Lambda^1)\\
&= \int(\star dA,\delta\star B)\,\omega_g \qquad (\delta = (-1)^{n-1}\star d\,\star \text{ in } \Lambda^{n-1})\\
&= \int(d\star dA,\star B)\,\omega_g \qquad (\text{Stokes formula})\\
&= \int(\star\delta dA,\star B)\,\omega_g \qquad (d\star = \star\delta \text{ in } \Lambda^2)\\
&= \int(\star\delta dA)\wedge(\star\star B) \qquad (\text{since } \alpha\wedge(\star\beta) = (\alpha,\beta)\omega_g \text{ in } \Lambda^p)\\
&= (-1)^{n-1}\int(\star\delta dA)\wedge B \qquad (\star\star = (-1)^{n-1} \text{ in } \Lambda^1)\\
&= \int(\delta dA,B)\,\omega_g \qquad (\alpha\wedge\beta = (-1)^{n-1}\beta\wedge\alpha \text{ for } \alpha\in\Lambda^{n-1},\beta\in\Lambda^1)
\end{aligned}$$

Thus,

$$\frac{1}{2}\left(\frac{d}{dA}\int|\nabla\times A|^2\right).(B) = \int\left(\overline{\Delta}_g A,B\right)$$

for all B, where $\overline{\Delta}_g = \delta d$, δ the codifferential, d the differential.

4. The Proca contribution is necessary in the closed setting

The Proca addition is essential in the closed setting (compact manifolds without boundaries) as the third equations in $(BPSP)_a$ and (SPP) would imply that $u\equiv 0$ if $m_1 = 0$, then that $\varphi\equiv C^{te}$ and that $A\equiv 0$ is trivial if the Ricci curvature of the manifold is positive (without the positive Ricci curvature assumption A has to be harmonic). By the Bochner-Lichnerowicz-Weitzenböck formula for 1-forms,

$$(\Delta_g A,A) = \frac{1}{2}\Delta_g|A|^2 + |\nabla A|^2 + \mathrm{Rc}_g(A^\sharp,A^\sharp)\ , \tag{4.1}$$

where $A^\sharp$ is the vector field we get from A by the musical isomorphism. The following elementary result holds true.

Lemma 4.1. *If (u,φ,A) is a solution of the m_1-free version of (2.1), namely of (2.1) with $m_1 = 0$, then $u = 0$, φ is constant and also, A is zero when the Ricci curvature of the manifold is positive.*

Proof of Lemma 4.1. Integrating the third equation in (2.1) gives $u\equiv 0$. Then, if we multiply the third equation in (2.1) by $\Delta_g\varphi$ and integrate over M we get that $\Delta_g\varphi\equiv 0$, and thus that φ is a constant. By the second equation in (2.1), $\overline{\Delta}_g A = \Delta_g A$. Contracting the fourth equation in (2.1) by $\Delta_g A$ and integrating over M, we get by the Bochner-Lichnerowicz-Weitzenböck formula (4.1) for the 1-form $\Delta_g A$ that

$$a^2\int_M|\nabla\Delta_g A|^2 dv_g + \int_M|\Delta_g A|^2 dv_g + a^2\int_M Rc_g\left((\Delta_g A)^\sharp,(\Delta_g A)^\sharp\right)dv_g = 0\ ,$$

where $(\Delta_g A)^\sharp$ is the vector field we get from $\Delta_g A$ by the musical isomorphism. When $Rc_g > 0$ in the sense of bilinear forms this implies that $\Delta_g A \equiv 0$, and since there are no harmonic 1-forms when the Ricci curvature is positive (another immediate consequence of the Bochner-Lichnerowicz-Weitzenböck formula), we get that $A \equiv 0$ is trivial. $\square$

5. What about Maxwell and the gauge invariance

There is a link between the Maxwell-Schrödinger-Proca system (1.5) and the classical Maxwell equations in modern format. We define the electric field E, the magnetic induction H, the charge density ρ and the current density J by the equations

$$\begin{aligned} &E = -\frac{1}{4\pi}\left(\frac{\partial A}{\partial t} + \nabla\varphi\right)\ ,\ H = \frac{1}{4\pi}\nabla \times A\ ,\\ &\rho = qu^2\ ,\ J = \frac{\hbar q}{m_0^2}\left(\nabla S - \frac{q}{\hbar}A\right)u^2\ . \end{aligned}$$

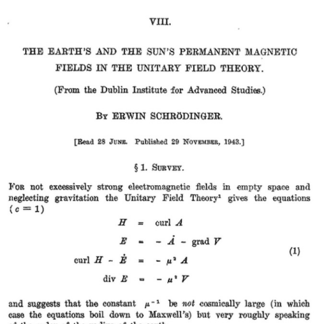
Since $\overline{\Delta}_g = \nabla \times \nabla \times$, the two last equations in (1.5) rewrite as

$$\begin{aligned} &\nabla.E + \frac{m_1^2}{4\pi}\varphi = \rho\ ,\\ &\nabla \times H - \frac{\partial E}{\partial t} + \frac{m_1^2}{4\pi}A = J \end{aligned}$$

and thus they rewrite as the first pair of the Maxwell-Proca equations with respect to a matter distribution whose charge and current density are respectively ρ and J. As usual, we get for free that the second pair of the equations holds true. Then the two last equations in (1.5) can be rewritten in the form of the massive modified Maxwell equations in SI units

$$\begin{aligned} &\nabla.E = \rho/\varepsilon_0 - \mu^2\varphi\ ,\\ &\nabla \times H = \mu_0\left(J + \varepsilon_0\frac{\partial E}{\partial t}\right) - \mu^2 A\ ,\\ &\nabla \times E + \frac{\partial H}{\partial t} = 0\ ,\ \nabla.H = 0\ , \end{aligned} \tag{5.1}$$

where, here, $\varepsilon_0 = 1$, $\mu_0 = 1$ and $\mu^2 = \frac{m_1^2}{4\pi}$. Such equations were discussed in Schrödinger [32].



VIII.

THE EARTH'S AND THE SUN'S PERMANENT MAGNETIC FIELDS IN THE UNITARY FIELD THEORY.

(From the Dublin Institute for Advanced Studies.)

By ERWIN SCHRÖDINGER.

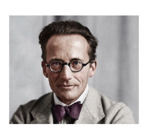
[Read 28 June. Published 29 November, 1943.]

§ 1. Survey.

For not excessively strong electromagnetic fields in empty space and neglecting gravitation the Unitary Field Theory[1] gives the equations ($c = 1$)

$$\begin{aligned} H &= \operatorname{curl} A\\ E &= -\dot{A} - \operatorname{grad} V\\ \operatorname{curl} H - \dot{E} &= -\mu^2 A\\ \operatorname{div} E &= -\mu^2 V \end{aligned} \tag{1}$$

and suggests that the constant μ^{-1} be *not* cosmically large (in which case the equations boil down to Maxwell's) but very roughly speaking of the order of the radius of the earth.



Erwin Schrödinger
1887 – 1961

The Earth's and the Sun's Permanent Magnetic Fields in the Unitary Field Theory

Erwin Schrödinger

Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences

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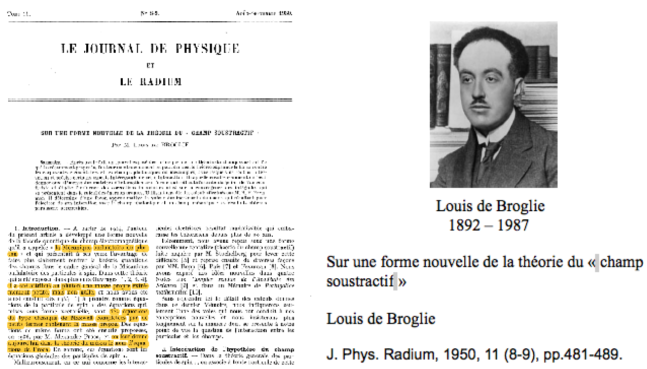
They also have been discussed by several other physicists. In addition to Proca and Schrödinger we could name De Broglie, Pauli, Yukawa, Stueckelberg. . . The point in these theories is that m_1 is nothing but the mass of the photon and we are therefore talking about a theory where photons have a mass. Recall that "the photon is the quantum of the electromagnetic field including electromagnetic radiation such as light, and the force carrier for the electromagnetic force" (Wikipedia). Physicists speak also of W bosons when the particles are massive (the W boson has mass approximatively 81 Gev, which means that it weights as 81 protons). More on this can be found in the survey papers by Gillies, Luo and Tu [27] and by Goldhaber and Nieto [17, 18]. We refer also to Adelberger-Dvali-Gruzinov [1] and Spallicci [33, 34].



Underlined text: . . . la Mécanique ondulatoire du photon . . .
il a été attribué au photon une masse propre extrêmement petite,
mais non nulle . . . des équations du type classique de Maxwell
complétées par des petits termes contenant la masse propre. . . . on leur
donne aujourd'hui dans la théorie du méson, le nom d'équations de Proca.

The first equation in (1.5) is a nonlinear Schrödinger equation. The second equation in (1.5) is the charge continuity equation $\frac{\partial\rho}{\partial t} + \nabla.J = 0$. This equation turns out to be equivalent to the Lorenz condition

$$\nabla.A + \frac{\partial\varphi}{\partial t} = 0 \tag{5.2}$$

when $m_1 \neq 0$ (and thus as soon as there is a nonzero Proca mass). This is easily seen by taking the derivative in time of the first equation in (5.1) and the divergence of the second equation in (5.1). In doing so we get that

$$\frac{\partial\rho}{\partial t} + \nabla.J = \mu^2\left(\nabla.A + \frac{\partial\varphi}{\partial t}\right) .$$

In other words, the condition $m_1 \neq 0$ (which turns out to be equivalent to $\mu \neq 0$ since $4\pi\mu^2 = m_1^2$) breaks the gauge invariance and enforces the Lorenz gauge. When φ is static we get the Coulomb gauge condition $\delta A = 0$ from (5.2).

6. A short discussion on Bopp-Podolsky-Proca

We very briefly discuss the Bopp-Podolsky-Proca equations in vaccum. More on the Maxwell-Proca and the Bopp-Podolsky models can be found in Cuzinatto-De Morais-Medeiros-Naldoni de Souza-Pimentel [11] and Zayats [43]. We adopt their notations here. Several other references are possible. The equations for the Maxwell-Proca electrodynamics in vacuum with Lorenz condition are

$$\Box\Phi + m^2\Phi = 0 \ , \tag{6.1}$$

where Φ represents the full field consisting of φ and A, and $\Box$ is the d'Alembert operator. This is exactly what we get with the two last equations in (1.5) with the unit $c = 1$, $m = m_1$ and when we cancel the u-terms in these equations (our convention on Δ_g makes that in the case of the Euclidean metric we get $-\Delta$, where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$). Remember, see above, that the Lorenz condition gives that $\partial_t\varphi = \delta A$. Then equation (6.1) describes photons with (small) mass m. The equations for the Bopp-Podolsky electrodynamics in vacuum are

$$a^2\Box^2\Phi + \Box\Phi = 0 \ . \tag{6.2}$$

The equations in the case of Bopp-Podolsky-Proca are

$$a^2\Box^2\Phi + \Box\Phi + m^2\Phi = 0 \ . \tag{6.3}$$

The traditional interpretation for (6.2) is that the equation splits into two second order equations

$$\begin{aligned} &\Box\hat{\Phi} = 0 \ , \\ &\Box\tilde{\Phi} + \frac{1}{a^2}\tilde{\Phi} = 0 \ , \end{aligned} \tag{6.4}$$

where $\hat{\Phi} = a^2\Box\Phi + \Phi$ and $\tilde{\Phi} = a^2\Box\Phi$. These two equations give two kinds of photons. The first equation in (6.4) describes massless photons and the second equation in (6.4) describes massive photons (with mass of the order of $1/a$). A theory with massless and massive photons requires fourth order equations, and as far as Bopp-Podolsky is involved, $a > 0$ is small. A similar interpretation can be given for (6.3). Define

$$\begin{cases} \hat{\Phi} = \Box\Phi + \frac{1+\sqrt{\Delta}}{2a^2}\Phi \\ \tilde{\Phi} = \Box\Phi + \frac{1-\sqrt{\Delta}}{2a^2}\Phi \ , \end{cases}$$

where $\Delta = 1 - 4a^2m^2$. Then

$$\begin{cases} \Box\hat{\Phi} + \frac{1-\sqrt{\Delta}}{2a^2}\hat{\Phi} = 0 \\ \Box\tilde{\Phi} + \frac{1+\sqrt{\Delta}}{2a^2}\tilde{\Phi} = 0 \ . \end{cases} \tag{6.5}$$

In this situation, we recover photons with "small" mass of the order of m by the first equation in (6.5), and massive photons with mass of the order of $1/a$ by the second equation in (6.5), the point here being that

$$\frac{1-\sqrt{\Delta}}{2a^2} \simeq m^2 \text{ and } \frac{1+\sqrt{\Delta}}{2a^2} \simeq \frac{1}{a^2}$$

as $a \to 0^+$. Given $\delta > 0$, if we let $m = m_a$ with

$$m_a^2 = \delta(1 - \delta a^2) \ , \tag{6.6}$$

then
$$\frac{1-\sqrt{\Delta}}{2a^2} = \delta \text{ and } \frac{1+\sqrt{\Delta}}{2a^2} = \frac{1-a^2\delta}{a^2}$$
and we get photons with mass $\sqrt{\delta}$ and massive photons with mass of the order of $1/a$. As a remark, one can choose to get two massive photons by letting $m = m_a$ with m_a as in (6.6), but now with $\delta = \delta_a$, $\delta_a \to +\infty$ and $a^2\delta_a \leq \frac{1}{2}$ for instance.

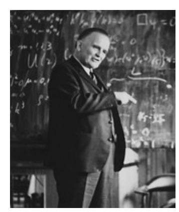
7. Short Biographies

(The biographies are from Wikipedia.)

In Romania, Alexandru Proca was one of the eminent students at the Gheorghe Lazăr High School and the Politehnica University in Bucharest. With a very strong interest in theoretical physics, he went to Paris where he graduated in Science from the Paris-Sorbonne University, receiving from the hand of Marie Curie his diploma of Bachelor of Science degree. After that he was employed as a researcher/physicist at the Radium Institute in Paris in 1925. He carried out Ph.D. studies in theoretical physics under the supervision of Nobel laureate Louis de Broglie. He defended successfully his Ph.D. thesis entitled "On the relativistic theory of Dirac's electron" in front of an examination committee chaired by the Nobel laureate Jean Perrin. In 1929, Proca became the editor of the influential physics journal *Les Annales de l'Institut Henri Poincaré*. Then, in 1934, he spent an entire year with Erwin Schrödinger in Berlin, and visited for a few months with Nobel laureate Niels Bohr in Copenhagen where he also met Werner Heisenberg and George Gamow.

Alexandru Proca
1897-1955

Boris Podolsky
1896-1966



Fritz Bopp
1909-1987

In 1896, Boris Podolsky was born into a poor Jewish family in Taganrog, in the Don Host Oblast of the Russian Empire, and he moved to the United States in 1913. After receiving a Bachelor of Science degree in Electrical Engineering from the University of Southern California in 1918, he served in the US Army and then worked at the Los Angeles Bureau of Power and Light. In 1926, he obtained an MS in Mathematics from the University of Southern California. In 1928, he received a PhD in Theoretical Physics (under Paul Sophus Epstein) from Caltech. Under a National Research Council Fellowship, Podolsky spent a year at the University of California, Berkeley, followed by a year at Leipzig University. In 1930, he returned to Caltech, working with Richard C. Tolman for one year. He then went to the Ukrainian Institute of Physics and Technology (Kharkiv, USSR), collaborating with Vladimir Fock, Paul Dirac (who was there on a visit), and Lev Landau. In 1932 he published a seminal early paper on Quantum Electrodynamics with Dirac and Fock. In 1933, he returned to the US with a fellowship from the Institute for Advanced Study, Princeton. In a letter dated November 10, 1933, to Abraham

Flexner, founding Director of the Institute for Advanced Study at Princeton, Einstein described Podolsky as "one of the most brilliant of the younger men who has worked and published with Dirac." In 1935, Podolsky took a post as professor of mathematical physics at the University of Cincinnati. In 1961, he moved to Xavier University, Cincinnati, where he worked until his death in 1966.

Friedrich Arnold "Fritz" Bopp was a German theoretical physicist who contributed to nuclear physics and quantum field theory. He worked at the Kaiser-Wilhelm Institut für Physik and with the Uranverein. He was a professor at the Ludwig Maximilian University of Munich and a President of the Deutsche Physikalische Gesellschaft. He signed the Göttingen Manifesto. From 1929 to 1934, Bopp studied physics at the Goethe University Frankfurt and the University of Göttingen. He completed his Diplom thesis in 1933 under the mathematician Hermann Weyl. In 1934, he became an Assistant at Göttingen. In 1937, Bopp completed his doctorate on the subject of Compton scattering under the physicist Fritz Sauter. From 1936 to 1941, he was a teaching assistant at Breslau University. In 1941, Bopp completed his Habilitationsschrift under Erwin Fues on the subject of a consistent field theory of the electron. From 1941 to 1947, Bopp was a staff scientist at the Kaiser-Wilhelm Institut für Physik (KWIP, after World War II reorganized and renamed the Max Planck Institute for Physics), located in Berlin-Dahlem. From 1946 to 1947, Bopp was also a teaching assistant at the University of Tübingen. From 1947 to 1950, Bopp was an extraordinarius professor and in 1950 an ordinarius professor of theoretical physics at the Institute of Theoretical Physics of the Ludwig Maximilian University of Munich. His main area of interest was quantum field theory. In 1954, he was a member of the board of trustees of the Institute. During 1956 and 1957, Bopp was a member of the Arbeitskreis Kernphysik (Nuclear Physics Working Group) of the Fachkommission II Forschung und Nachwuchs (Commission II Research and Growth) of the Deutschen Atomkommission (DAtK, German Atomic Energy Commission). From 1964 to 1965, Bopp was the President of the Deutsche Physikalische Gesellschaft.

8. The results we obtained on $(BPSP)_a$ and (SPP)

The Maxwell-Proca and Bopp-Podolsky-Proca models that we couple with the Schrödinger equation (in the electro-static and electro-magneto-static cases) were investigated in the case of closed manifolds in Hebey [20, 21, 22], Hebey and Wei [23] and Thizy in the series of papers [37, 38, 39, 40, 41]. We refer also to Azzollini-d'Avenia-Pomponio [6], d'Avenia, Mederski and Pomponio [4], d'Avenia and Siciliano [5], Benci-Fortunato [7, 8, 9], Figueiredo-Siciliano [15], Ianni [24] and Ianni and Vaira [25] for these equations when the ground space is the Euclidean space and (in almost all of these papers) the Proca mass is set to zero. This list is far from being exhaustive.

We concentrate here on electro-magneto-static solutions to our equations, in the case of closed 3-manifolds, and thus we concentrate on the reduced equations $(BPSP)_a$ and (SPP) of the introduction. This mainly concerns (in this very specific context) the papers Hebey [20, 21, 22]. There is a notion of critical exponent for Sobolev embeddings. In dimension 3 the critical exponent is 6. This explains the restriction $p \leq 6$. The equations are subcritical when $p < 6$ and critical precisely when $p = 6$. We also assume that $p \geq \frac{22}{5}$ (though there are situations where we can go down at least to 4). A coercive operator like $\Delta_g + \Lambda_g$ has positive mass (resp. nonnegative mass) if the regular part of its Green's function is positive (resp. nonnegative) on the diagonal. By the positive mass theorem of Schoen and Yau [31] (see also Witten [42]), there exists a function Λ_g (with $\Lambda_g > 0$ in M) such that $\Delta_g + \Lambda_g$ has positive mass when the scalar curvature S_g of the manifold is positive. A condition we will use in the critical case of the exponent is that

$$\omega^2 + \frac{\hbar^2}{2m_0^2}|\nabla S|^2 < \frac{\hbar^2}{2m_0^2}\Lambda_g \tag{8.1}$$

in M, where $\Lambda_g > 0$ is smooth and such that $\Delta_g + \Lambda_g$ has nonnegative mass. In the case of the standard 3-sphere we can take $\Lambda_g = \frac{3}{4}$, and this is the best possible value. We discuss four questions here:

(1) Existence of a solution to our systems,
(2) Robustness of our systems with respect to variations of the parameters,
(3) Convergence of the Bopp-Podolsky-Proca system $(BPSP)_a$ to the Schrödinger-Poisson-Proca system (SPP) as the Bopp-Podolsky parameter $a \to 0$.
(4) Collapsing of the Bopp-Podolsky-Proca system $(BPSP)_a$ to the sole Schrödinger equation as the Bopp-Podolsky parameter $a \to 0$.

We present five theorems below. Theorem A answers the first question. Theorems B and C answer the second question. Theorem D answers the third question and Theorem E answers the fourth question. We start with the answers to the first question about existence. We let Rc_g be the Ricci curvature of g.

Theorem A (Existence for $(BPSP)_a$ and (SPP))**.** *Let (M, g) be a smooth closed 3-manifold, $\omega \in \mathbb{R}$, $a \in \mathbb{R}^+$ be a nonnegative real number, $q, m_0, m_1 > 0$ be positive real numbers and $S \in C^\infty_R(M)$ be a smooth real-valued function. Let $p \in [\frac{22}{5}, 6]$. We assume that $Rc_g + m_1^2 g > 0$ in the sense of bilinear forms when $a = 0$, that $am_1 < \frac{1}{2}$ and when p is critical from the viewpoint of Sobolev embeddings, namely when $p = 6$, we also assume that* (8.1) *holds true. Then both $(BPSP)_a$ when $a > 0$, and (SPP) when $a = 0$, possess a smooth nontrivial solution (u, v, A) with $u > 0$ and $v > 0$ in M. Also $A \not\equiv 0$ when $\nabla S \not\equiv 0$.*

Theorem A is proved in Hebey [20] and [22]. It leaves a question open: *find a solution which includes the Coulomb gauge condition $\delta A = 0$.* A specific answer (corresponding to a special choice of S) in the case of the Euclidean space is given in Benci-Fortunato [8] and d'Avenia, Mederski and Pomponio [4].

Going on, passing to the second question, robustness of the systems with respect to variations of the parameters is evaluated in terms of the notion of stability which has been intensively discussed in book form in Hebey [19]. Let $(a_\alpha)_\alpha$, $(m_\alpha)_\alpha$, $(\omega_\alpha)_\alpha$ be sequences of real numbers and $(S_\alpha)_\alpha$ be a sequence of functions. Given α integer we define

$$(BPSP)_\alpha \overset{def}{=} (BPSP)_{a_\alpha} \text{ when } (\omega_\alpha, m_\alpha, S_\alpha) \text{ is in place of } (\omega, m_1, S)\ ,$$

$$(SPP)_\alpha \overset{def}{=} (SPP) \text{ when } (\omega_\alpha, m_\alpha, S_\alpha) \text{ is in place of } (\omega, m_1, S)\ .$$

In other words we get two sequences of systems. For any α, $(SPP)_\alpha$ is like (SPP) when ω is replaced by ω_α, m_1 is replaced by m_α and S is replaced by S_α, while $(BPSP)_\alpha$ is like $(BPSP)_a$ when a is replaced by a_α, ω is replaced by ω_α, m_1 is replaced by m_α and S is replaced by S_α. In some occasions we could have also authorized p to vary (e.g. see Hebey [20]). What we refer to as strong robustness here is what was referred to as bounded stability in Hebey [19]. Obviously, strong robustness implies compactness of the equations. Concerning Theorem B, but also Theorems C, D and E, the $(BPSP)_a$'s and (SPP)'s equations could have been replaced by $(\overline{BPSP})_a$ and $(\overline{SPP})$ since the Coulomb gauge condition is preserved under the convergences we do prove for the A_α's. As a general remark for Theorems B, C, D and E, a (nonnegative) solution u to the first equation in $(BPSP)_a$ or (SPP) is either everywhere 0 or everywhere positive (an easy consequence of the maximum principle).

Theorem B (Strong Robustness for $(BPSP)_a$)**.** *Let (M,g) be a smooth closed 3-manifold, $\omega \in \mathbb{R}$, $a, q, m_0, m_1 > 0$ be positive real numbers and $S \in C^\infty_R(M)$ be a smooth real-valued function. Let $p \in [\frac{22}{5}, 6]$. When p is critical from the viewpoint of Sobolev embeddings, namely when $p = 6$, we assume that* (8.1) *holds true. Then $(BPSP)_a$ is strongly robust with respect to variations of its coefficients in the sense that for any sequence $(a_\alpha)_\alpha$ of real numbers converging to a, for any sequence $(\omega_\alpha)_\alpha$ of real numbers converging to ω, for any sequence $(m_\alpha)_\alpha$ of real numbers converging to m_1, for any sequence $(S_\alpha)_\alpha$ of smooth real valued functions converging in $C^{1,\theta}_R$ to S for some $\theta \in (0,1)$ and for any sequence $((u_\alpha, v_\alpha, A_\alpha))_\alpha$ of solutions of $(BPSP)_\alpha$, there holds that, up to passing to a subsequence, $u_\alpha \to u$ in C^2_R, $v_\alpha \to v$ in C^2_R, $A_\alpha \to A$ in C^2_V and (u,v,A) solve $(BPSP)_a$. When $am_1 < \frac{1}{2}$ there also holds that $u > 0$ and $v > 0$ in M if $(u_\alpha)_\alpha$ is nontrivial. Also $A \not\equiv 0$ when $u > 0$ and $\nabla S \not\equiv 0$.*

Theorem B is proved in Hebey [20]. We did not have the a_α's and m_α's in [20] (the a_α's were fixed to a and the m_α's were fixed to $m_1 > 0$) but this variation implies no essential changes in the proof (since the a_α's here, contrary to what is discussed in Theorem D, stay far from zero).

Theorem C (Strong Robustness for (SPP))**.** *Let (M,g) be a smooth closed 3-manifold, $\omega \in \mathbb{R}$, $q, m_0, m_1 > 0$ be positive real numbers and $S \in C^\infty_R(M)$ be a smooth real-valued function. Let $p \in [\frac{22}{5}, 6]$. We assume that $Rc_g + m_1^2 g > 0$ in the sense of bilinear forms and when p is critical from the viewpoint of Sobolev embeddings, namely when $p = 6$, we also assume that* (8.1) *holds true. Then (SPP) is strongly robust with respect to variations of its coefficients in the sense that for any sequence $(\omega_\alpha)_\alpha$ of real numbers converging to ω, for any sequence $(m_\alpha)_\alpha$ of real numbers converging to m_1, for any sequence $(S_\alpha)_\alpha$ of smooth real valued functions converging in $C^{1,\theta}_R$ to S for some $\theta \in (0,1)$ and for any sequence $((u_\alpha, v_\alpha, A_\alpha))_\alpha$ of solutions of $(SPP)_\alpha$, there holds that, up to passing to a subsequence, $u_\alpha \to u$ in C^2_R, $v_\alpha \to v$ in C^2_R, $A_\alpha \to A$ in C^2_V and (u,v,A) solve (SPP). Moreover $u > 0$ and $v > 0$ in M if $(u_\alpha)_\alpha$ is nontrivial. Also $A \not\equiv 0$ when $u > 0$ and $\nabla S \not\equiv 0$.*

Theorem C is proved in Hebey [22]. We did not have the m_α's in [22] (the m_α's were fixed to $m_1 > 0$) but, here again, this variation implies no essential changes in the proof. Theorems B and C might seem natural to a non expert. This is true in the subcritical case, but not in the critical case which often generates unstable solutions. In this specific context one might refer to the blowing-up examples in Hebey and Wei [23]. Suppose $M = S^3$. It is proved there that there exists an increasing sequence $(\omega_k)_{k\geq 1}$ of phases such that $\omega_1 = \frac{\sqrt{3}\hbar}{2\sqrt{2}m_0}$, such that $\omega_k \to +\infty$ as $k \to +\infty$, and such that both all the $-\omega_k$'s and ω_k's are unstable in the sense that: for any k, there exists a sequence $(\omega_\alpha)_\alpha$ of real numbers such that $\omega_\alpha^2 \to \omega_k^2$ as $\alpha \to +\infty$, and there exist sequences $(u_\alpha)_\alpha$ and $(v_\alpha)_\alpha$ of functions satisfying that

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u_\alpha + \omega_\alpha^2 u_\alpha + q v_\alpha u_\alpha = u_\alpha^{p-1} \\ \Delta_g v_\alpha + m_1^2 v_\alpha = 4\pi q u_\alpha^2 \end{cases} \tag{8.2}$$

for all $\alpha \in \mathbb{N}$, with the property that $\|u_\alpha\|_{L^\infty_R} \to +\infty$ as $\alpha \to +\infty$ (and therefore also $\|v_\alpha\|_{C^2_R} \to +\infty$ as $\alpha \to +\infty$) and the additional property that the u_α's exhibit k single isolated bumps in their blow-up processes. This equation (8.2) is nothing but $(SPP)_\alpha$ when $m_\alpha = m_1$, $A_\alpha \equiv 0$ and $S_\alpha \equiv 0$ for all α. Also $\Lambda_0 = \frac{\sqrt{3}\hbar}{2\sqrt{2}m_0}$ is nothing but than the square root of the quantity in the RHS of (8.1) in the case of S^3. In other words, resonant frequencies appear outside $(-\Lambda_0, +\Lambda_0)$, starting with $\pm\Lambda_0$, and the threshold Λ_0 is critical for ω.

Going on, passing to the third question, we want to discuss what happens to $(BPSP)_a$ as $a \to 0$. There we are asking whether or not a fourth order system converges in a strong sense to a second order system. We slightly change the definition of $(BPSP)_\alpha$ for Theorems D and E. For $(a_\alpha)_\alpha$ a sequence of positive real numbers, and $(m_\alpha)_\alpha$ another sequence of positive real numbers, we let

$$(BPSP)_\alpha \overset{def}{=} (BPSP)_{a_\alpha} \text{ when } m_\alpha \text{ is in place of } m_1 .$$

In other words, we do not touch to ω and S in Theorems D and E below (we could have) and $(BPSP)_\alpha$ is like $(BPSP)_a$ when a is replaced by a_α and m_1 is replaced by m_α. The point of course in Theorems D and E is that we aim to send $a_\alpha \to 0$. In what follows H^2_R (resp. H^2_V) stands for the Sobolev space of functions (resp. 1-forms) with two derivatives in L^2. By Sobolev, $H^2 \subset C^{0,\theta}$ for all $\theta \in (0,1)$.

Theorem D (Strong convergence of $(BPSP)_a$ to (SPP) as $a \to 0$)**.** *Let (M,g) be a smooth closed 3-manifold, $\omega \in \mathbb{R}$, $q, m_0, m_1 > 0$ be positive real numbers and $S \in C^\infty_R(M)$ be a smooth real-valued function. Let $p \in [\frac{22}{5}, 6]$. We assume that $Rc_g + m_1^2 g > 0$ in the sense of bilinear forms and when p is critical from the viewpoint of Sobolev embeddings, namely when $p = 6$, we also assume that (8.1) holds true. Then for any sequence $(a_\alpha)_\alpha$ of positive real numbers converging to zero, any sequence $(m_\alpha)_\alpha$ of positive real numbers converging to m_1, and any sequence $((u_\alpha, v_\alpha, A_\alpha))_\alpha$ of solutions of $(BPSP)_\alpha$, there holds that, up to passing to a subsequence, $(u_\alpha, v_\alpha, A_\alpha) \to (u, v, A)$ in $C^{2,\theta}_R \times H^2_R \times H^2_V$ as $\alpha \to +\infty$, where (u, v, A) is a solution of (SPP), with $u > 0$ and $v > 0$ in M as soon as $(u_\alpha)_\alpha$ is nontrivial, and also $A \not\equiv 0$ as soon as $(u_\alpha)_\alpha$ is nontrivial and $\nabla S \not\equiv 0$.*

Theorem D describes a noncollapsing situation. Another reference where such a convergence (in the case of specific solutions) is obtained is d'Avenia and Siciliano [5]. Theorem E below addresses on the contrary the collapsing case. We let (S) be the Schrödinger equation

$$\frac{\hbar^2}{2m_0^2}\Delta_g u + \Phi_0 u = u^{p-1} , \tag{S}$$

where $\Phi_0 = \omega^2 + \frac{\hbar^2}{2m_0^2}|\nabla S|^2$. In Theorem D, $m_\alpha \to m_1$ as $\alpha \to +\infty$, where $m_1 > 0$, while in Theorem E, $m_\alpha \to +\infty$. The two situations could correspond to the equations

$$m_\alpha^2 = \frac{1-\sqrt{1-4a_\alpha^2 m_1^2}}{2a_\alpha^2} \text{ and } m_\alpha^2 = \frac{1+\sqrt{1-4a_\alpha^2 m_1^2}}{2a_\alpha^2}$$

as they are discussed in Section 6.

Theorem E (Collapsing of $(BPSP)_a$ to (S) as $a \to 0$). *Let (M,g) be a smooth closed 3-manifold, $\omega \in \mathbb{R}$, $a, q, m_0 > 0$ be positive real numbers and $S \in C_R^\infty(M)$ be a smooth real-valued function. Let $p \in [\frac{22}{5}, 6]$. When p is critical from the viewpoint of Sobolev embeddings, namely when $p = 6$, we assume that (8.1) holds true. Then for any sequence $(a_\alpha)_\alpha$ of positive real numbers converging to zero, for any sequence $(m_\alpha)_\alpha$ of positive real numbers converging to $+\infty$ and for any sequence $((u_\alpha, v_\alpha, A_\alpha))_\alpha$ of solutions of $(BPSP)_\alpha$, there holds that , up to passing to a subsequence, $(u_\alpha, v_\alpha, A_\alpha) \to (u, 0, 0)$ in $C_R^{2,\theta} \times H_R^2 \times H_V^2$ as $\alpha \to +\infty$, where u is a solution of (S). Moreover $u > 0$ in M if $(u_\alpha)_\alpha$ is nontrivial and either $\omega \neq 0$ or $\nabla S \not\equiv 0$.*

9. Few (very few) words on the proofs

We very briefly discuss the proofs of Theorems A and B in the case of $(BPSP)_a$ and present the outline of the analysis in a series of steps that are more and more intricate. We may here assume $p > 4$. Let H_R^k be the Sobolev space of functions in L^2 with k derivatives in L^2 and H_V^k be the corresponding space for 1-forms.

Level 1: Functional analysis.1

The first result one can prove is that for any $u \in H_R^1$, there exists a unique $A(u) \in H_V^4 \cap C_V^2$ such that

$$a^2\Delta_g^2 A(u) + \Delta_g A(u) + \left(m_1^2 + \frac{4\pi q^2}{m_0^2}u^2\right)A(u) = \frac{4\pi q\hbar}{m_0^2}(\nabla S)u^2 \ .$$

Then there also holds that there exist $C, C' > 0$ such that

$$\|A(u)\|_{H_V^2} \le C\|u\|_{L_R^2}\min(1, \|u\|_{L_R^2})$$

and

$$\|A(u)\|_{H_V^4} \le C''\left(1+\|u\|_{L_R^4}^2\right)\|u\|_{L_R^4}$$

for all $u \in H_R^1$. In addition the map $A : H_R^1 \to H_V^2$ is locally Lipschitz and differentiable with A'_u given by

$$\begin{aligned} &a^2\Delta_g^2 A'_u(h) + \Delta_g A'_u(h) + \left(m_1^2 + \frac{4\pi q^2}{m_0^2}u^2\right)A'_u(h) \\ &= \frac{8\pi q\hbar}{m_0^2}u\left(\nabla S - \frac{q}{\hbar}A(u)\right)h \ , \ h \in H_R^1 \ . \end{aligned}$$

Let $\mathcal{I}_1 : H_R^1 \to \mathbb{R}$ be given by

$$\mathcal{I}_1(u) = \int_M \left(\nabla S - \frac{q}{\hbar}A(u), \nabla S\right)u^2 dv_g \ . \tag{9.1}$$

Thanks to the above we get that $\mathcal{I}_1$ is differentiable and

$$\mathcal{I}_1'(u).(v) = 2\int_M \left|\nabla S - \frac{q}{\hbar}A(u)\right|^2 uhdv_g \tag{9.2}$$

for all $u, v \in H_R^1$. All this can be proved using variational analysis and elliptic regularity type arguments.

Level 2: Functional analysis.2

The second result one can prove is that for any $u \in H_R^1$, there exists a unique $v(u) \in H_R^4 \cap C_R^2$ such that

$$a^2\Delta_g^2 v(u) + \Delta_g v(u) + m_1^2 v(u) = 4\pi q u^2 \ .$$

Then we can also prove that there exist $C, C' > 0$ such that the following two estimates hold. Namely,

$$\|v(u)\|_{H_R^2} \le C\|u\|_{L_R^2}^2 \ , \text{ and}$$
$$\|v(u)\|_{H_R^4} \le C'\|u\|_{L_R^4}^2$$

for all $u \in H_R^1$. Moreover $v : H_R^1 \to H_R^2$ is locally Lipschitz and there also holds that $v : H_R^1 \to H_R^2$ is differentiable with the property that for any $u \in H_R^1$ its differential $v'_u \in L(H_R^1, H_R^2)$ is given by

$$a^2\Delta_g^2 v'_u(h) + \Delta_g v'_u(h) + m_1^2 v'_u(h) = 8\pi q u h$$

for all $h \in H_R^1$. Let $\mathcal{I}_2 : H_R^1 \to \mathbb{R}$ be given by

$$\mathcal{I}_2(u) = \int_M v(u)u^2 dv_g \ . \tag{9.3}$$

Then $\mathcal{I}_2$ is differentiable and

$$\mathcal{I}'_2(u).(v) = 4\int_M v(u)uv dv_g \tag{9.4}$$

for all $u, v \in H_R^1$. Here again all this can be proved using variational analysis and elliptic regularity type arguments.

Level 3: Functional analysis.3

Another result we need is the following. This is where the condition

$$am_1 < \frac{1}{2}$$

comes into the story. Let $v : H_R^1 \to H_R^2$ be the map in Level 2. Assuming that $am_1 < \frac{1}{2}$ there holds that $v(u) \ge 0$ for all $u \in H_R^1$ and that there exists $\varepsilon_0 > 0$, independent of u, such that

$$\int_M \left(|\nabla u|^2 + v(u)u^2\right) dv_g \ge \varepsilon_0 \|u\|_{H_R^1}^2 \min\left(1, \|u\|_{H_R^1}^2\right) \tag{9.5}$$

for all $u \in H_R^1$. We easily get the existence of $a_1, a_2, a_3, a_4 > 0$ such that

$$a^2\Delta_g^2 + \Delta_g + m_1^2 = (a_1\Delta_g + a_2)(a_3\Delta_g + a_4) \ .$$

if $\kappa = \frac{a_1}{a_2}$ satisfies that

$$m_1^2\kappa^2 - \kappa + a^2 = 0 \ .$$

The discriminant is given by

$$\Delta = 1 - 4a^2m_1^2 \ .$$

The condition $am_1 < \frac{1}{2}$ guarantees the splitting.

Thanks to Levels 1 to 3 the problem has a variational structure that we can handle. The variational structure is going to be useful on what concerns the existence of a solution, but not so much on what concerns compactness.

Level 4: Functional setting

The following result directly follows from Levels 1 to 3. Let $2 < p \leq 6$. Assume that $am_1 < \frac{1}{2}$. Define $I_p : H^1_R \to \mathbb{R}$ to be the functional

$$\begin{aligned} I_p(u) &= \frac{\hbar^2}{4m_0^2}\int_M |\nabla u|^2 dv_g + \frac{\omega^2}{2}\int_M u^2 dv_g \\ &+ \frac{q}{4}\int_M v(u)u^2 dv_g + \frac{\hbar^2}{4m_0^2}\int_M \left(\nabla S - \frac{q}{\hbar}A(u), \nabla S\right) u^2 dv_g \\ &- \frac{1}{p}\int_M (u^+)^p dv_g \ , \end{aligned}$$

where A and v are as in Levels 1 and 2, and where $u^+ = \max(0,u)$. Then I_p is differentiable and if u is a critical point of I_p, on can prove that $(u, v(u), A(u))$ is a smooth solution of $(BPSP)_a$ with $u, v \geq 0$.

Level 5: Existence of a solution

We prove existence of a solution in the subcritical case by using level 4 and the mountain pass lemma of Ambrosetti-Rabinowitz [2]. Basically if one starts low, then has to climb a mountain, and goes down again, then he gets a Palais-Smale sequence for his functional. Basically the energy converges and the derivative of the energy goes to zero. By compactness of the embeddings, which is given for free in the subcritical case, we get a critical point at the limit.

In the critical case we use the MPL combined with the test functions introduced by Schoen in his resolution of the 3-dimensional Yamabe problem. Given $u_0 \in H^1_R$, define

$$c(u_0) = \inf_{P\in\mathcal{P}} \max_{u\in P} I_6(u) \ ,$$

where $\mathcal{P}$ denotes the class of continuous paths joining 0 to u_0 and I_6 is the functional I_p in the critical case $p = 6$. Picking u_0 as a Schoen's test function, assuming the critical condition (8.1), there exists $\delta_0 > 0$ such that

$$\delta_0 \leq c(u_0) \leq \frac{1}{3K_3^3} - \delta_0 \ ,$$

where K_3 is the sharp constant in the Euclidean Sobolev inequality. Then the Aubin-Brézis-Nirenberg arguments work for MPL. We are below the best constant and we recover compactness. As in the subcritical case we then get a critical point in the limit.

Level 6: Stability

Let (M, g) be a smooth closed 3-manifold, $\omega \in \mathbb{R}$, $a, q, m_0, m_1 > 0$ be positive real numbers and $S \in C^\infty_R(M)$ be a smooth real-valued function. Let $(a_\alpha)_\alpha$ be a sequence converging to a, $(m_\alpha)_\alpha$ be a sequence converging to m_1, $(\omega_\alpha)_\alpha$ be sequence converging to ω, $p \in (4, 6]$ and $(S_\alpha)_\alpha$ be a sequence in C^∞_R which converges to S

in $C^{1,\theta}_R$ as $\alpha \to +\infty$ for some $\theta \in (0,1)$. Assume the critical condition (8.1) when $p = 6$. Let $(u_\alpha, v_\alpha, A_\alpha)_\alpha$, $u_\alpha > 0$, be a sequence of solutions of

$$\begin{cases} \frac{\hbar^2}{2m_0^2}\Delta_g u_\alpha + \Phi_\alpha(x, v_\alpha, A_\alpha)u_\alpha = u_\alpha^{p_\alpha - 1} \\ a^2\Delta_g^2 v_\alpha + \Delta_g v_\alpha + m_1^2 v_\alpha = 4\pi q u_\alpha^2 \\ a^2\Delta_g^2 A_\alpha + \Delta_g A_\alpha + m_1^2 A_\alpha = \frac{4\pi q\hbar}{m_0^2}\Psi(A_\alpha, S_\alpha)u_\alpha^2 \ . \end{cases} \tag{9.6}$$

We aim to prove that, up to passing to a subsequence, $u_\alpha \to u$, $v_\alpha \to v$ in C^2_R and $A_\alpha \to A$ in C^2_V as $\alpha \to +\infty$, for some $u, v \in C^2_R$ and $A \in C^2_V$ which solve $(BPSP)_a$ with the additional property that if $am_1 < \frac{1}{2}$, then we also have that $u > 0$ and $v > 0$ in M. We will be very sketchy here. The proof is quite involved. We start with some control estimate on the u_α's. We divide the first equation by u_α and integrate. There holds that $\int \frac{\Delta u}{u} \leq 0$ and by the estimates in Levels 1 and 2 there holds that

$$\int \Phi_\alpha \leq C(1 + \|u_\alpha\|^2_{L^2_R}) \ .$$

Then

$$\int_M u_\alpha^{p_\alpha - 2} dv_g \leq C\left(1 + \|u_\alpha\|^2_{L^2_R}\right)$$

for all α, where $C > 0$ is independent of α. Then, since $p > 4$, the u_α's are bounded in L^2_R and with this little control on the u_α's we can prove that the Φ_α's are bounded in $C^{0,\theta}_R$ for some $\theta \in (0,1)$. Then the proof of the stability in the subcritical case $p < 6$ essentially follows the Gidas-Spruck [16] arguments based on elementary blow-up and the fact that the equation $\Delta u = u^{p-1}$ does not have any nontrivial solution in $\mathbb{R}^3$ when p is subcritical.

In the critical case $p = 6$ the situation is much more involved, the point being that the equation $\Delta u = u^5$ now has plenty of solutions in $\mathbb{R}^3$ (the extremals for the Sobolev inequality, see Aubin [3] and Talenti [36]). We use here the 3-dimensional particularity (this essentially goes back to Schoen [30], Li-Zhu [26], Marques [28], Druet [12, 13]) that blow-up points are always isolated. A priori we could have cluster's type configurations for the u_α's (groups of bubbles interacting one with another). It turns out that, in dimension 3, such clusters never occur and we always have in dimension 3 configurations with a repetition of single bumps. Then the blow-up analysis implies that we do have bounded energy and we recover a well defined H^1-Struwe [35] type decomposition for the u_α's, meaning by this that we recover well defined blow-up points for the u_α's. Moreover, it turns out (as we just mentioned) that these blow-up points are isolated.

Coming back to Levels 1 and 2 we get more control on the Φ_α's and we can branch on the Schoen [30], Li-Zhu [26], Druet [12, 13], Marques [28], Druet-Hebey-Robert [14] ... advanced analysis for blow-up which gives C^2-convergence as soon as the limit operator for $\frac{\hbar^2}{2m_0^2}\Delta_g + \Phi_\alpha$ is coercive and has positive mass (a reference in book form is Hebey [19]). The limit operator here is

$$\frac{\hbar^2}{2m_0^2}\Delta_g + \Phi \ ,$$

where $\Phi = \frac{\hbar^2}{2m_0^2}|\nabla S|^2 + \omega^2$. This is clearly a coercive operator when $\omega \neq 0$ or S is not constant. By the critical condition in the theorem

$$\frac{2m_0^2}{\hbar^2}\Phi < \Lambda_g \ ,$$

and since Λ_g has nonnegative mass, the maximum principle gives that the limit operator has positive mass. When $\omega = 0$ and S is constant, we use a blow-up property of the Green's function $G_\alpha : M \times M \backslash D \to \mathbb{R}$ of $\frac{\hbar^2}{2m_0^2}\Delta_g + \Phi_\alpha$, namely that G_α blows up in the sense that $\inf_{M\times M\backslash D} G_\alpha \to +\infty$ as $\alpha \to +\infty$. Then, again, we can conclude with arguments as in Hebey and Wei [23].

When $a \to 0$ in $(BPSP)_a$ many of the estimates above are lost and we need to rebuild the whole theory on $A(u)$ and $v(u)$ and make the estimates independent of a. This leads to new difficulties. The analysis is carried over in Hebey [21].

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Emmanuel Hebey, Université de Cergy-Pontoise, Département de Mathématiques, Site de Saint-Martin, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France
E-mail address: `Emmanuel.Hebey@cyu.fr`