Elliptic stability for stationary Schrödinger equations

by

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Part VI/VI

Applications

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Nonlinear analysis arising from geometry and physics

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PART VI. APPLICATIONS TO OUR ORIGINAL EQUATIONS.

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NOTE : The blue writing is what you have to write down to be able to follow the slides presentation.

VI.1) The Klein-Gordon-Maxwell-Proca system :

Let (M,g) be a closed manifold of dimension $n \geq 3$. Let $m_0, m_1 > 0$ and $q > 0$. Let $\omega \in (-m_0, +m_0)$, and $p \in (2, 2^\star]$. We consider the electrostatic KGMP reduced system

$$\begin{cases} \Delta_g u + m_0^2 u = u^{p-1} + \omega^2 (qv - 1)^2 u \\ \Delta_g v + \left(m_1^2 + q^2 u^2\right) v = qu^2 \ . \end{cases} \qquad (S_\omega)$$

We assume $m_1 > 0$ (Proca formalism). If not the case, $v = \frac{1}{q}$ and the two equations are independent one from another. If $\left(u_\alpha e^{-i\omega_\alpha t}\right)_\alpha$ and $(v_\alpha)_\alpha$ solve our system, then

$$\begin{cases} \Delta_g u_\alpha + m_0^2 u_\alpha = u_\alpha^{p-1} + \omega_\alpha^2 (qv_\alpha - 1)^2 u_\alpha \\ \Delta_g v_\alpha + \left(m_1^2 + q^2 u_\alpha^2\right) v_\alpha = qu_\alpha^2 \ . \end{cases} \qquad (S_\alpha)$$

The soliton family $\left(u_\alpha e^{-i\omega_\alpha t}\right)_\alpha$ has finite energy if $\|u_\alpha\|_{H^1} = O(1)$. Let $S_p(\omega) = \left\{\left(ue^{-i\omega t}, v\right),\ u, v > 0 \text{ smooth, which solve } (S_\omega)\right\}$, and for $\mathcal{U} = \left(ue^{-i\omega t}, v\right)$, let also $\|\mathcal{U}\|_{C^{2,\theta}} = \|u\|_{C^{2,\theta}} + \|v\|_{C^{2,\theta}}$.

Definition (a priori bound, stable phase, resonant state)

Let (M, g) be a smooth compact Riemannian manifold of dimensions $n \geq 3$. Let $m_0, m_1 > 0$, $q > 0$, and $p \in (2, 2^\star]$. Let $\omega \in (-m_0, +m_0)$. We say that

(i) ω gives rise to the a priori bound property if there exist $\varepsilon > 0$ and $C > 0$ such that $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$ for all $\mathcal{U} \in S_p(\tilde{\omega})$ and all $\tilde{\omega} \in (\omega - \varepsilon, \omega + \varepsilon)$,

(ii) ω is a stable phase if for any sequence $\left(u_\alpha e^{-i\omega_\alpha t}\right)_\alpha$ of finite energy standing waves, and any sequence $(v_\alpha)_\alpha$ of gauge electric fields, solutions of (S_α), the convergence $\omega_\alpha \to \omega$ in $\mathbb{R}$ implies that, up to a subsequence, $u_\alpha \to u$ and $v_\alpha \to v$ in C^2, where $ue^{-i\omega t}$ and v solve (S_ω).

At last we say that ω is a resonant phase, or give rise to resonant states, if there exist a sequence $\left(u_\alpha e^{-i\omega_\alpha t}\right)_\alpha$ of finite energy standing waves, and a sequence $(v_\alpha)_\alpha$ of gauge electric fields, solutions of (S_α), s.t. $\omega_\alpha \to \omega$ and $\|u_\alpha\|_{L^\infty} + \|v_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$.

A priori bound property $\Rightarrow$ phase stability property

(Bounded stability property) $\Rightarrow$ (Analytic stability property)

Variational structure, natural energy functional :

$$S(u,v) = \frac{1}{2}\int_M |\nabla u|^2 dv_g - \frac{\omega^2}{2}\int_M |\nabla v|^2 dv_g + \frac{m_0^2}{2}\int_M u^2 dv_g$$
$$-\frac{\omega^2 m_1^2}{2}\int_M v^2 dv_g - \frac{1}{p}\int_M u^p dv_g - \frac{\omega^2}{2}\int_M u^2(1-qv)^2 dv_g \ .$$

Define $\Phi : H^1 \to H^1$ by

$$\Delta_g \Phi(u) + \left(m_1^2 + q^2 u^2\right)\Phi(u) = qu^2 \ .$$

We can prove that Φ is differentiable when $n = 3, 4$. Define I_p by

$$I_p(u) = \frac{1}{2}\int_M |\nabla u|^2 dv_g + \frac{m_0^2}{2}\int_M u^2 dv_g - \frac{1}{p}\int_M (u^+)^p dv_g$$
$$-\frac{\omega^2}{2}\int_M \left(1 - q\Phi(u)\right) u^2 dv_g \ .$$

The critical points of I_p are the solutions of (S_ω).

Goals : prove the existence of solutions to our systems, $^{(\star)}$ prove the a priori bound property, prove the phase stability property when the a priori bound property does not hold true, and prove the existence of resonant states when the phase stability property does not hold true.

In the subcritical case (analogue of the Gidas-Spruck theorem) :

Theorem 0 (Subcritical case ; Druet-H., 2010 ; H.Truong, 2012)

Let (M, g) be a smooth compact Riemannian n-dimensional manifold, $n \geq 3$, $m_0, m_1 > 0$, and $q > 0$. Let $p \in (2, 2^\star)$. For any $\omega \in (-m_0, +m_0)$ there exists a smooth positive mountain pass solution of (S_ω). Moreover, for any $\theta \in (0,1)$, there exists $C > 0$ such that $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$ for all $\mathcal{U} \in S_p(\omega)$ and all $\omega \in (-m_0, +m_0)$.

N.B. : Thm 0 prevents existence of standing waves with arbitrarily large amplitudes.

$^{(\star)}$: We look for variational solutions such as mountain pass solutions for I_p (ground states models in the Nehari-Rabinowitz sense).

Question : when $p = 2^\star$ what should we require on m_0, m_1, and ω in order to get a similar result ? What about resonant states ?

Theorem 1 (A priori bounds ; Druet-H., 2010 ; H.-Truong, 2012)

Let (M, g) be a smooth compact Riemannian manifold of dimensions $n = 3, 4$. Let $m_0, m_1 > 0$ and $q > 0$ be positive real numbers. Let $\omega \in (-m_0, +m_0)$ and $p = 2^\star$. Assume

$$m_0^2 < \omega^2 + \frac{n-2}{4(n-1)} S_g \tag{1}$$

in M, where S_g is the scalar curvature of g. Then (S_ω) possesses a smooth positive mountain pass solution. Moreover, there also holds that for any $\theta \in (0,1)$, there exists $C > 0$ such that $\|\mathcal{U}\|_{C^{2,\theta}} \leq C$ for all $\mathcal{U} \in S_{2^\star}(\omega')$ and all $\omega' \in (-m_0, +m_0)\backslash(-|\omega|, +|\omega|)$.

Concerning existence when $n = 4$ we just need to have (1) at one point in M. The problem is local in that case. When $n = 3$ we may replace the scalar curvature term by the maximum potential term for which we do have positivity of the mass.

Consequence 1 :

Whatever m_0 is, there exists $\varepsilon_0 > 0$ such that we do get existence and a priori bounds in the range $m_0^2 - \varepsilon_0 < \omega^2 < m_0^2$ (phase compensation).

Consequence 2 :

In case $m_0^2 < \frac{n-2}{4(n-1)} S_g$, we do get existence and a priori bounds for all phases, and thus for the full range of phases.

Here again we prevent the existence of standing waves with arbitrarily large amplitudes (e.g., when $m_0 \gg 1$, fast oscillating standing waves cannot have arbitrarily large amplitudes).

When $m_0 \gg 1$, Theorem 1 answers our question for large ω's, and we are left with the question for the other values of ω, namely when $\omega^2 \leq m_0^2 - \frac{n-2}{4(n-1)} S_g$. Here the answer depends strongly on the dimension.

Theorem 2 (3-dim resonant states ; H.-Wei, 2012)

Let $(\mathbb{S}^3, g)$ be the unit 3-sphere, $m_0, m_1 > 0$, and $q > 0$. Let $p = 2^\star$. There exists a sequence $(\theta_k)_k$ of positive real numbers, satisfying that $\theta_1 = \frac{\sqrt{3}}{2}$, $\theta_k > \theta_1$ when $k \geq 2$, and $\theta_k \to +\infty$ as $k \to +\infty$, and there exists a sequence $(c_k(m_1))_k$, satisfying that $c_1(m_1) = 0$, $c_k(m_1) > 0$ for $k \geq 2$, and $c_k(m_1) \to +\infty$ as $k \to +\infty$, such that any $\omega_k \in (-m_0, m_0)$ given by $\theta_k^2 = m_0^2 - \omega_k^2$, which satisfy that $q^2\omega_k^2 \neq c_k(m_1)$, is an resonant phase for (S_ω) associated with a k-spikes configuration.

For any such ω_k, there exists $(u_\alpha e^{-i\omega_\alpha t})_\alpha$ and $(v_\alpha)_\alpha$ solutions of

$$\begin{cases} \Delta_g u_\alpha + m_0^2 u_\alpha = u_\alpha^{2^\star - 1} + \omega_\alpha^2 \left(q v_\alpha - 1\right)^2 u_\alpha \\ \Delta_g v_\alpha + \left(m_1^2 + q^2 u_\alpha^2\right) v_\alpha = q u_\alpha^2 \end{cases} \qquad (S_\alpha)$$

for all α, such that $\omega_\alpha \to \omega_k$ and $\|u_\alpha\|_{L^\infty} + \|v_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$ (and k bubbles are involved in the construction).

The condition $q^2\omega_k^2 \neq c_k(m_1)$ is automatically satisfied when we require that $q m_0 \ll m_1$.

Theorem 3 (4-dimensional phase stability ; Druet-H.-Vétois, 2013)

Let (M,g) be a smooth compact Riemannian manifold of dimension $n=4$. Let $m_0, m_1 > 0$ and $q > 0$ be positive real numbers. Let $\omega \in (-m_0, +m_0)$, and $p = 2^\star$. Assume

$$m_0^2 - \omega^2 \not\in \left[\frac{1}{6}\min_M S_g, \frac{1}{6}\max_M S_g\right]$$

Then ω is a stable phase for (S_ω). Conversely, on the standard sphere $(\mathbb{S}^4, g)$, when $m_0^2 \geq \frac{1}{6}\max_M S_g$, the two $\pm\omega$'s given by the equation $m_0^2 - \omega^2 = \frac{1}{6}S_g$ are resonant phases for (S_ω).

N.B. : the first part of the result holds true even when S_g is not positive. In particular all phases are stable when $S_g \leq 0$ in M (like in the model cases of flat torii and compact hyperbolic spaces).

The first part of the theorem is false when $n = 3$ by the preceding theorem (establishing the existence of a whole family of resonant states for small ω's).

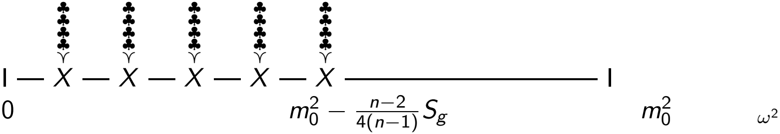
Summarizing in the $\mathbb{S}^3$ and $\mathbb{S}^4$ model cases :

[1] Case of $\mathbb{S}^3$: $\left(\frac{n-2}{4(n-1)} S_g = \frac{3}{4}\right)$

Resonant States$^{\text{Thm2}}$

A priori bounds$^{\text{Thm1}}$

(No resonant states)



[2] Case of $\mathbb{S}^4$: $\left(\frac{n-2}{4(n-1)} S_g = 2\right)$

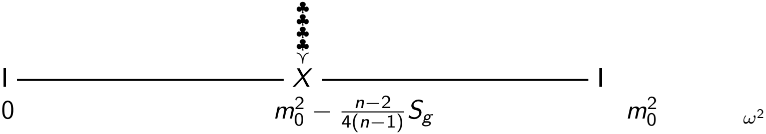
Phase stability$^{\text{Thm3}}$

(No resonant states)

A priori bounds$^{\text{Thm1}}$

(No resonant states)

Resonant states$^{\text{Thm3}}$



VI.2) The Einstein-Lichnerowicz equation :

Let (M, g) be a closed manifold of dimension $n \geq 3$. Consider the constraint system of general relativity in the conformal method setting :

$$\begin{cases} \Delta_g u + hu = fu^{2^\star - 1} + \frac{a}{u^{2^\star + 1}} \\ \overrightarrow{\Delta}_g X = \frac{n-1}{n} u^{\frac{2n}{n-2}} \nabla \tau + \pi \nabla \psi \ , \end{cases} \qquad \text{(ELCE)}$$

where u, X are the unknowns and h, f, a are given by

$$h = \frac{n-2}{4(n-1)} \left(S_g - |\nabla \psi|^2 \right) \ ,$$

$$f = \frac{n-2}{4(n-1)} \left(2V(\psi) - \frac{n-1}{n} \tau^2 \right) \ ,$$

$$a = \frac{n-2}{4(n-1)} \left(|\sigma + \mathcal{L}_g X|^2 + \pi^2 \right) \ .$$

The conformal method, starting from given physics data $(\psi, \pi, \tau, \sigma)$ and V, generates initial data sets for the original initial constraint system of equations by the parametrization rule

$$(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi}) = \left(u^{\frac{4}{n-2}} g, \frac{\tau}{n} u^{\frac{4}{n-2}} g + u^{-2}(\sigma + \mathcal{L}_g X), \psi, u^{-\frac{2n}{n-2}} \pi \right) . \quad \text{(Par)}$$

Using the Choquet-Bruhat-Geroch theorem (CBG), we get space-time developments solutions of the Einstein equations. This is the Choquet-Bruhat-Geroch-Lichnerowicz (CBGL) formalism which we summarize here :

Freely chosen physics data $(\psi, \pi, \tau, \sigma)$ and V

$\downarrow$ Step 1 : Solving (ELCE)

Solution(s) (u, X) of (ELCE)

$\downarrow$ Step 2 : (Par)

Initial data set(s) $\left(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi}\right)$

$\downarrow$ Step 3 : CBG-Thm

Maximal space-time development(s) $(\mathcal{M}, h, \Psi)$

Question : *is the CBGL formalism robust with respect to the initial choice of the physics data $(\psi, \pi, \tau, \sigma)$ and V of the conformal method ?*

Existence for CMC focusing case ($\tau = C^t$, $f > 0$) : Hebey-Pacard-Pollack (2008). Robustness for CMC focusing case : Druet-Hebey (2009). Existence for the full system in the focusing case : Premoselli (2014).

In the 3-dimensional case, the robustness question for the full system in the focusing case is due to Druet and Premoselli (2014). The more general case is due to Premoselli (2015).

Theorem 4 (Stability of the CBGL formalism ; Premoselli 2015)

Let (M, g) be a closed locally conformally flat Riemannian manifold of dimension $n \geq 3$. Assume that the data are focusing ($f > 0$) and that $\pi \not\equiv 0$. If $n \geq 6$, assume in addition that τ and ψ have no common critical points in M. Let $(V_\alpha)_\alpha$ and $(D_\alpha)_\alpha$, $D_\alpha = (\psi_\alpha, \pi_\alpha, \tau_\alpha, \sigma_\alpha)_\alpha$ be sequences of potentials and of physics data converging respectively to V and D in the following topology :

$$\|V_\alpha - V\|_{C^2} + \|\tau_\alpha - \tau\|_{C^3} + \|\psi_\alpha - \psi\|_{C^2} + \|\pi_\alpha - \pi\|_{C^0} + \|\sigma_\alpha - \sigma\|_{C^0} \underset{\alpha \to +\infty}{\longrightarrow} 0.$$

Consider $(u_\alpha, X_\alpha)_\alpha$ a sequence of solutions of the Einstein-Lichnerowicz constraints system with physics data D_α and V_α. Then, up to a subsequence and up to conformal Killing 1-forms, the sequence $(u_\alpha, X_\alpha)_\alpha$ converges in $C^{1,\eta}(M)$, for any $0 < \eta < 1$, to some solution (u_0, X_0) of the limiting Einstein-Lichnerowicz constraints system of equations with physics data D and V. In particular, the CBGL formalism is stable with respect to the choice of generic focusing initial data $(\psi, \pi, \tau, \sigma)$ and V in any locally conformally flat geometry in M.

VI.3) The Kirchhoff equation :

Let (M, g) be a closed manifold of dimension $n \geq 3$, $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ be two sequences of positive real numbers, and $(h_\alpha)_\alpha$ be a sequence of C^1-functions $h_\alpha : M \to \mathbb{R}$. Consider

$$\left(a_\alpha + b_\alpha \int_M |\nabla u|^2 dv_g\right) \Delta_g u + h_\alpha u = u^{2^\star - 1} \qquad (\mathrm{K}_\alpha)$$

A sequence $(u_\alpha)_\alpha$ is said to be a sequence of nonnegative solutions of (K_α) if the u_α's are nonnegative and solve the α-equation (K_α) for any α.

We always assume in the sequel that the a_α's and b_α's converge in $\mathbb{R}$, and that the h_α's converge in C^1. We regard such (K_α)'s as perturbations of the orginal Kirchhoff system (K) :

$$\left(a + b \int_M |\nabla u|^2 dv_g\right) \Delta_g u + hu = u^{2^\star - 1} \qquad (\mathrm{K})$$

Both (K) and (K_α) have a variational structure.

The (K_α)'s come with $I_\alpha : H^1 \to \mathbb{R}$ given by

$$I_\alpha(u) = \frac{a_\alpha}{2}\int_M |\nabla u|^2 dv_g + \frac{b_\alpha}{4}\left(\int_M |\nabla u|^2 dv_g\right)^2 + \frac{1}{2}\int_M h_\alpha u^2 dv_g - \frac{1}{2^\star}\int_M |u^+|^{2^\star} dv_g \ .$$

Let $(u_\alpha)_\alpha$ be a sequence in H^1. As before we say that the sequence $(u_\alpha)_\alpha$ is a Palais-Smale sequence for $(I_\alpha)_\alpha$ if : (i) the $I_\alpha(u_\alpha)$'s are bounded, (ii) and $I'_\alpha(u_\alpha) \to 0$ in $(H^1)'$ as $\alpha \to +\infty$.

The H^1-theory for the blow-up applies here : for any PS-sequence $(u_\alpha)_\alpha$ of nonnegative functions, up to passing to a subsequence,

$$u_\alpha = u_\infty + \sum_{i=1}^k M_\alpha^{\frac{1}{2^\star - 2}} B_\alpha^i + R_\alpha \qquad (H^1\text{Dec}) \ ,$$

where, u_∞ solves (K), k is an integer, the $(B_\alpha^i)_\alpha$'s are bubbles, the R_α's converge strongly to 0 in H^1, and the M_α's, which come from the nonlocal aspects of the equations, are given by

$$M_\alpha = a_\alpha + b_\alpha \int_M |\nabla u_\alpha|^2 dv_g \ .$$

The sequence $(u_\alpha)_\alpha$ blows up if $k \geq 1$. We define

$$\mathcal{N}(u_\alpha) = \max\left\{k \text{ in } (H^1\text{Dec}) \text{ for subsequences of } (u_\alpha)_\alpha\right\} ,$$

the maximal number of bubbles we can have in H^1-dec. of subsequences of the u_α's.

Theorem 5 : (H.-Thizy, 2014)

Let (M, g) be a closed Riemannian 3-manifold, $a, b > 0$ be positive real numbers, and $h : M \to \mathbb{R}$ be a C^1-function. For any sequences $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ of positive real numbers converging to a and b, any sequence $(h_\alpha)_\alpha$ of C^1-functions $h_\alpha : M \to \mathbb{R}$ converging C^1 to h, and any sequence $(u_\alpha)_\alpha$ of nonnegative solutions of (K_α), there holds that $\|u_\alpha\|_{H^1} = O(1)$. Moreover, if $\Delta_g + \frac{1}{a}h$ is coercive, and the u_α's blow up, then

$$a + bK_3^{-3}\sqrt{C}\mathcal{N}(u_\alpha) \leq C ,$$

where K_3 is the sharp constant in the Euclidean Sobolev inequality, $\mathcal{N}(u_\alpha)$ is as above, and $C > 0$ depends only on (M, g, h) through the inequality $h \leq C\Lambda_g$ in M, where $\Lambda_g > 0$ is such that $\Delta_g + \Lambda_g$ has positive mass.

The following corollary holds true.

Corollary : (H.-Thizy, 2014)

Let (M, g) be a closed Riemannian 3-manifold, and $\Lambda_g > 0$ be such that $\Delta_g + \Lambda_g$ has positive mass. Let $a, b > 0$ be positive real numbers, and $h : M \to \mathbb{R}$ be a C^1-function such that $\Delta_g + \frac{1}{a}h$ is coercive. Assume that

$$h(x) < \left(a + \frac{1}{2}b^2 K_3^{-6} + \frac{1}{2}bK_3^{-3}\sqrt{4a + b^2 K_3^{-6}}\right)\Lambda_g(x)$$

for all $x \in M$. Then (K) has a nonnegative nontrivial C^2-solution. Moreover, for any $\theta \in (0, 1)$, there exists $C > 0$ such that $\|u_\alpha\|_{C^{2,\theta}} \leq C$ for all sequences $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ converging to a and b, all sequences $(h_\alpha)_\alpha$ of C^1-functions $h_\alpha : M \to \mathbb{R}$ converging C^1 to h, and all sequences $(u_\alpha)_\alpha$ of nonnegative solutions of (K_α).

In the higher dimensional case, let's assume that

$$h \equiv \frac{n-2}{4(n-1)} S_g \ , \qquad (\star)$$

where S_g scalar curvature of g.

Theorem 6 : (H.-Thizy, 2014)

Let (M, g) be a closed Riemannian n-manifold with positive scalar curvature, $n = 4$ or 5, $a, b > 0$ be positive real numbers, and $h : M \to \mathbb{R}$ be given by the geometric model $(\star)$. Assume that

$$\frac{1-a}{b} \notin K_n^{-n} \mathbb{N}^\star \ ,$$

where K_n is the sharp Sobolev constant. For any $\theta \in (0,1)$, there exists $C > 0$ s.t. $\|u_\alpha\|_{C^{2,\theta}} \leq C$ for all $(a_\alpha)_\alpha$, $(b_\alpha)_\alpha$ converging to a and b, all $(h_\alpha)_\alpha$ in $C^1(M, \mathbb{R})$ converging C^1 to h, and all sequences $(u_\alpha)_\alpha$ of nonnegative solutions of (K_α).

More results : Let (M,g) be a closed Riemannian n-manifold, $n \geq 4$, $a, b > 0$, and $h \in C^1(M,\mathbb{R})$ be s.t. $\Delta_g + \frac{1}{a}h$ is coercive. Assume one of the following assumptions :

(1) (H^1-compactness) a and b satisfy that $bK_n^{-n}a^{\frac{n-4}{2}} > \frac{2}{n-2}\left(\frac{n-4}{n-2}\right)^{\frac{n-4}{2}}$ when $n \geq 5$, and $bK_4^{-4} > 1$ when $n = 4$,

(2) (positive geometries) $S_g > 0$ everywhere in M, and

$$h(x) < \frac{(n-2)a}{4(n-1)}\left(1 + bK_n^{-n}a^{\frac{n-4}{2}}\right)S_g(x)$$

for all $x \in M$,

(3) (nonpositive geometries) $h(x) > 0$ for all $x \in M$, $S_g \leq 0$ everywhere in M, and $n = 5$ or $n \geq 7$,

where S_g is the scalar curvature of g, and K_n is the sharp Sobolev constant. Then for any $\theta \in (0,1)$, there exists $C > 0$ such that $\|u_\alpha\|_{C^{2,\theta}} \leq C$ for all $(a_\alpha)_\alpha$, $(b_\alpha)_\alpha$ converging to a and b, all $(h_\alpha)_\alpha$ in $C^1(M,\mathbb{R})$ converging C^1 to h, and all sequences $(h_\alpha)_\alpha$ of nonnegative solutions of (K_α). + Existence of nonnegative (nontrivial) solutions in cases (1) and (2).

VI.4) Proof of Theorem 5 based on the bounded stability theorem :

Here $n = 3$. We want to prove that sequences $(u_\alpha)_\alpha$ of solutions of (K_α) are bounded in H^1 and that the number k of bubbles we can have in H^1-decompositions of such sequences is bounded from above by

$$a + bK_3^{-3}\sqrt{C}k \leq C \ ,$$

where $C > 0$ is such that $h \leq C\Lambda_g$, and $\Lambda_g > 0$ is such that $\Delta_g + \Lambda_g$ has positive mass. The proof is typical of the 3-dimensional blow-up analysis. Let $(u_\alpha)_\alpha$ be a sequence of nonnegative nontrivial solutions of (K_α). We use the 3-dimensional blow-up machinery and get that

$$\begin{aligned} 3 - \text{dim. blow-up machinery} \Rightarrow \ & \text{Blow-up points are isolated} \\ \Rightarrow \ & \text{the } u_\alpha\text{'s are bounded in } H^1. \end{aligned}$$

Then we can assume that $M_\alpha \to M_\infty$ as $\alpha \to +\infty$. Still by the 3-dimensional blow-up analysis we get that there need to be a point where the mass of the vectorial Schrödinger operator $M_\infty \Delta_g + h$ is nonpositive. By comparison principles this implies that $\frac{1}{M_\infty} h$ can't be less than Λ_g.

In other words,

$$3-\text{dim. blow-up machinery (again)} \Rightarrow \exists x \in M \text{ s.t.} h(x) \geq M_\infty \Lambda_g(x) \\ \text{and } u_\infty \equiv 0.$$

Of course we recover the H^1-decomposition of the u_α's since the u_α's are bounded in H^1. In 3-space dimension, $\int_M |\nabla B_\alpha|^2 dv_g = K_3^{-3} + o(1)$. By the splitting of the energy associated with (H^1Dec), and since $u_\infty \equiv 0$,

$$\begin{aligned} M_\alpha &\overset{def}{=} a_\alpha + b_\alpha \int_M |\nabla u_\alpha|^2 dv_g \\ &= a_\alpha + b_\alpha \left(\sqrt{M_\alpha} k K_3^{-3} + o(1) \right) . \end{aligned}$$

Passing to the limit $\alpha \to +\infty$, $M_\infty = a + bk\sqrt{M_\infty} K_3^{-3}$, and then

$$\sqrt{M_\infty} = \frac{bkK_3^{-3} + \sqrt{b^2k^2K_3^{-6} + 4a}}{2} .$$

There exist s $x \in M$ such that $h(x) \geq M_\infty \Lambda_g(x)$. By assumption $h \leq C\Lambda_g$. Then $M_\infty \leq C$, and we easily get that

$$a + bK_3^{-3}\sqrt{C}k \leq C .$$

This is exactly what Theorem 5 says. ■

VI.5) Proof of the corollary :

We want to prove that if $h : M \to \mathbb{R}$ is such that $\Delta_g + \frac{1}{a} h$ is coercive and

$$h(x) < \left(a + \frac{1}{2} b^2 K_3^{-6} + \frac{1}{2} b K_3^{-3} \sqrt{4a + b^2 K_3^{-6}} \right) \Lambda_g(x)$$

for all $x \in M$, where $\Lambda_g > 0$ is a positive function such that $\Delta_g + \Lambda_g$ has positive mass, then

(i) the Kirchhoff system (K) has a nonnegative nontrivial C^2-solution,

(ii) $\forall \theta \in (0,1)$, $\exists C > 0$ such that $\|u_\alpha\|_{C^{2,\theta}} \leq C$ for all sequences $(a_\alpha)_\alpha$ and $(b_\alpha)_\alpha$ converging to a and b, all sequences $(h_\alpha)_\alpha$ of C^1-functions $h_\alpha : M \to \mathbb{R}$ converging C^1 to h, and all sequences $(u_\alpha)_\alpha$ of nonnegative solutions of (K_α).

The proof of (i) and (ii) is based on Theorem 5 showing that Theorem 5 remains valid if we replace the $2^\star$-exponent in (K_α) by subcritical exponents $p_\alpha \leq 2^\star$, $p_\alpha \to 2^\star$ (and this is true by the possible extension of the bounded stability theorem to asymptotically critical subcritical exponents).

We consider perturbations like

$$\left(a_\alpha + b_\alpha \int_M |\nabla u|^2 dv_g\right)\Delta_g u + h_\alpha u = u^{p_\alpha - 1} , \qquad (\tilde{K}_\alpha)$$

where $a_\alpha \to a$, $b_\alpha \to b$, $h_\alpha \to h$ in C^1, and $p_\alpha \le 2^\star$, $p_\alpha \to 2^\star$ as $\alpha \to +\infty$. Theorem 5 remains true in this context : for any sequence $(u_\alpha)_\alpha$ of nonnegative solutions of $(\tilde{K}_\alpha)$, the u_α's are bounded in H^1 and, up to a subsequence, the number k of bubbles they can have in their H^1-decomposition is s.t. $a + bK_3^{-3}\sqrt{C}k \le C$, where $C > 0$ is such that $h \le C\Lambda_g$. The subcritical equations always have nonnegative nontrivial solutions (variational arguments). By elliptic theory, it remains to prove that we can't have $k \ge 1$. In particular, the corollary holds true if $a + bK_3^{-3}\sqrt{C} > C$, and

$$\begin{aligned} & a + bK_3^{-3}\sqrt{C} > C \iff \\ & C < a + \frac{1}{2}b^2K_3^{-6} + \frac{1}{2}bK_3^{-3}\sqrt{4a + b^2K_3^{-6}} . \end{aligned}$$

The coercivity of $\Delta_g + \frac{1}{a}h$ implies that the limit profile $u_\infty \not\equiv 0$. This proves the corollary. ■

VI.6) Proof of Theorem 6 based on the bounded stability and the analytic stability theorems :

We mix here the two type of blow-up arguments : the bounded stability argument (to prove boundedness in H^1 when $n = 4$), and the analytic stability argument. Let $(a_\alpha)_\alpha$, $(b_\alpha)_\alpha$, $(h_\alpha)_\alpha$ be such that $a_\alpha \to a$, $b_\alpha \to b$, $h_\alpha \to h$ in C^1 as $\alpha \to +\infty$. Let $(u_\alpha)_\alpha$ be a sequence of nonnegative nontrivial solutions of (K_α). Recall we assume $S_g > 0$ in M. Suppose $M_\alpha \to +\infty$. Then, $\frac{h_\alpha}{M_\alpha} \to 0$, $0 < \frac{n-2}{4(n-1)} S_g$, and

$$\text{(Arg.1) Bounded stability theory} \Rightarrow \text{ Blow-up points are isolated}$$
$$\Rightarrow \text{ the } u_\alpha\text{'s are bounded in } H^1,$$

a contradiction ! In particular, the u_α's are bounded in H^1 and we get H^1-decompositions for the u_α's. Assume the u_α's blow up. Then

$$\text{(Arg.2) Analytic Stability theory } \Rightarrow \ u_\infty \equiv 0 \text{ (n}= 4,5)$$
$$\text{and } \exists\, x \in M \text{ s.t. } \frac{1}{M_\infty} h(x) = \frac{n-2}{4(n-1)} S_g(x)$$

where M_∞ is the limit of the M_α's defined as before.

By assumption

$$h \equiv \frac{n-2}{4(n-1)} S_g$$

and thus (by the Analytic Stability theory) we need to have that $M_\infty = 1$. H^1-decomposition (and $u_\infty \equiv 0$) imply that

$$\begin{aligned} M_\alpha &\stackrel{def}{=} a_\alpha + b_\alpha \int_M |\nabla U_\alpha|^2 dv_g \\ &= a + bkK_n^{-n} M_\alpha^{\frac{2}{2^\star - 2}} + o(1) \ . \end{aligned}$$

Then

$$M_\infty = a + bkK_n^{-n} M_\infty^{\frac{2}{2^\star - 2}} \ ,$$

and since $M_\infty = 1$, this implies that

$$\frac{1-a}{b} = K_n^{-n} k \ .$$

In other words, $\frac{1-a}{b} \in K_n^{-n} \mathbb{N}^\star$ if the u_α's blow up. This clearly proves Theorem 6. ∎

Thank you for your attention !