Elliptic stability for stationary Schrödinger equations
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Part IV/VI
Back to stability
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Nonlinear analysis arising from geometry and physics
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PART IV. Back to Stability.

NOTE : The blue writing is what you have to write down to be able to follow the slides presentation.

NOTE* : In what follows (E_h) refers to

$$\Delta_g u + hu = u^{2^\star - 1} , \qquad (E_h)$$

$u \geq 0$ in M.

PART IV. BACK TO STABILITY.

IV.1) Unstable situations :

The very first basic result is concerned with the Yamabe equation on the sphere. The Yamabe equation in S^n is written as

$$\Delta_g u + \frac{n(n-2)}{4} u = u^{2^\star - 1} \ , \qquad (Y_S)$$

and we know all its solutions (as by Caffarelli-Gidas-Spruck we know all the solutions of $\Delta u = u^{2^\star - 1}$ in $\mathbb{R}^n$). The solutions of (Y_S) are given by

$$u_{x_0,\beta}(x) = \left(\frac{n(n-2)}{4} (\beta^2 - 1) \right)^{\frac{n-2}{4}} (\beta - \cos r)^{1-\frac{n}{2}} \ ,$$

where $x_0 \in S^n$, $\beta > 1$ (including $\beta \to +\infty$), and $r = d_g(x_0, x)$. They all have the same energy :

$$\int_{S^n} u_{x_0,\beta}^{2^\star} dv_g = \frac{1}{K_n^n}$$

for all $x_0 \in S^n$ and all $\beta > 1$. There holds $u_{x_0,\beta} \to 0$ far from x_0. On the other hand, $\lim_{\beta \to 1} u_{x_0,\beta}(x_0) = +\infty$ since $u_{x_0,\beta}(x_0) \approx (\beta - 1)^{-(n-2)/4}$.

Theorem : (the sphere case)

There are sequences $(u_\alpha)_\alpha$ of solutions of the Yamabe equation (Y_S) on the sphere S^n such that $\|u_\alpha\|_{L^\infty} \to +\infty$ as $\alpha \to +\infty$.

It suffices to let $u_\alpha = u_{x_0,\beta_\alpha}$, where $(\beta_\alpha)_\alpha$ is a sequence such that $\beta_\alpha > 1$ and $\beta_\alpha \to 1$. Then we can show that they have an H^1-decomposition like $u_\alpha = B_\alpha + R_\alpha$, where $(B_\alpha)_\alpha$ is the bubble of centers $x_\alpha = x_0$ and weights $\mu_\alpha \approx \sqrt{\beta_\alpha - 1}$.

Gluing together such $u_{x_0,\beta}$'s we easily get more sophisticated blow-up configurations. By gluing we mean by hand constructions like

$$u_\alpha = \sum_{i=1}^{k} u_{x_{i,\alpha},\beta_\alpha} \ .$$

Making the gluing construction invariant under the action of groups the results we can prove extend to quotients of the sphere. Letting $k \to +\infty$, we get families of solutions with unbounded energy. These are by-hand constructions where we "naively" compute the potentials h_α for which the u_α's solve (E_{h_α}) and then try to find conditions on the $x_{i,\alpha}$'s and β_α's for which we will get a nice convergence of the h_α's. The following result holds true.

Theorem : (Space forms constructions, Druet-H., 2004)

Let $(S^n/G, g)$ be a space form of dimension $n \geq 6$.

(i) Let $1 \leq k_1 \leq k_2$ be given integers. There exist a sequence $(h_\alpha)_\alpha$ of smooth functions converging C^1 to $\frac{n(n-2)}{4}$, and a sequence $(u_\alpha)_\alpha$ of smooth positive functions solutions of

$$\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{2^\star - 1} , \qquad (E_\alpha)$$

in S^n/G for all α, such that the u_α's have bounded energy (namely $\|u_\alpha\|_{H^1} = O(1)$) and such that they blow up with k_2 bubbles in their H^1-decomposition and k_1 geometric blow-up points (the limits of the centers of the bubbles).

(ii) There also exist a sequence $(h_\alpha)_\alpha$ of smooth functions converging C^1 to $\frac{n(n-2)}{4}$, and a sequence $(u_\alpha)_\alpha$ of smooth positive solutions of (E_α) such that $\|u_\alpha\|_{H^1} \to +\infty$ as $\alpha \to +\infty$.

In case (i), picking $k_2 > k_1$ we get that there are bubbles accumulating on one single point (e.g., picking $k_1 = 1$ and $k_2 \geq 2$ then k_2 bubbles in the constructions accumulate to one single point). In case (ii), the sequence $(u_\alpha)_\alpha$ of solutions is unbounded in H^1.

This result can be refined by using the Lyapounov-Schmidt finite dimensional reduction method.

Theorem : (With the Lyapounov-Schmidt method)

(i) (Chen-Wei-Yan, 2011) Let (S^n, g) be the unit n-sphere, $n \geq 5$. For any $\lambda > \frac{n(n-2)}{4}$ there exists a sequence $(u_\alpha)_\alpha$ of positive solutions (E_λ) in S^n such that $\|u_\alpha\|_{H^1} \to +\infty$ as $\alpha \to +\infty$.
(ii) (Esposito-Pistoia-Vétois, 2013) Let (M, g) be a closed manifold of positive Yamabe invariant, $n \geq 4$, and $h \in C^{0,\theta}$ be such that $\max_M h > 0$. When $n \geq 6$ and g is not conformally flat we assume that there exists $c > 0$ such that the Weyl tensor W_g satisfies that $|W_g(x)| \geq c$ for all x where h is positive. Then there exists a sequence $(\varepsilon_\alpha)_\alpha$ of positive real numbers converging to zero, and a sequence $(u_\alpha)_\alpha$ of solutions of (E_{h_α}), $h_\alpha = \frac{n-2}{4(n-1)} S_g + \varepsilon_\alpha h$, such that $(u_\alpha)_\alpha$ blows up in a $B_\alpha + R_\alpha$ configuration (one bubble, one geometric blow-up point).
(iii) (Robert-Vétois, 2013) Let (M, g) be a closed non conformally flat manifold of positive Yamabe invariant, $n \geq 6$. Let $k \geq 1$, $r \geq 0$ two arbitrary integers. There exist sequences $(h_\alpha)_\alpha$ and $(u_\alpha)_\alpha$ such that $h_\alpha \to \frac{n-2}{4(n-1)} S_g$ in C^r and such that the $(u_\alpha)_\alpha$ blows up in a $\sum B_\alpha^i + R_\alpha$ configuration with one single geometric blow-up point (k bubbles, one geometric blow-up point).

This theorem improves the preceding theorem in several remarkable ways. In (i) we get unbounded energy solutions with a fixed potential. In (ii), we get blow-up on very general manifolds, up to $n \geq 4$, very explicit potentials h_α, and a C^∞-convergence of the potentials. In (iii), we get arbitrarily sophisticated blow-up configurations on very general manifolds, with arbitrarily high convergence of the potentials.

The theorem leaves open the case of dimension 3. Still by the use of the Lyapounov-Schmidt method, we can prove that :

Theorem : (The 3-dimensional case, H.-Wei, 2012)

Let (S^3, g) be the 3-sphere. There exists a sequence $(\theta_k)_k$ of positive real numbers such that $\theta_1 = \frac{3}{4}$, $\theta_k > \theta_1$ when $k \geq 2$, and $\theta_k \to +\infty$ as $k \to +\infty$, with the property that to each θ_k is associated a sequence $(\lambda_\alpha)_\alpha$ of positive real numbers converging to θ_k, and a sequence $(u_\alpha)_\alpha$ of positive solutions of (E_{λ_α}) such that $(u_\alpha)_\alpha$ blows up in a $\sum B_\alpha^i + R_\alpha$ configuration with k geometric blow-up points (k bubbles, k geometric blow-up points).

When $k = 1$, this is just the sphere case since $\frac{3}{4} = \frac{n(n-2)}{4} = \frac{n-2}{4(n-1)} S_g$ in (S^3, g). The result is sharp since (Bidaut-Véron and Véron) equation (E_λ) on the sphere has a sole constant solution for all $0 < \lambda < \frac{n(n-2)}{4}$.

IV.2) The Lyapounov-Schmidt method :

We very briefly discuss the Lypounov-Schmidt method. The method has been successfully used (in several problems) by several people among who (the list is far to be exhaustive) A.Bahri, S. Brendle, F. Coda-Marques, M. del Pino, Y. Ge, M. Kowalczyk, A. Malchiodi, R. Mazzeo, A.M.Micheletti, M. Musso, F. Pacard, F. Pacella, A. Pistoia, O.Rey, F. Robert, J. Vétois, J.Wei, etc.

The general idea is to obtain solutions of equations as perturbations of a given profile. Let $(W_{t,\alpha})_\alpha$ be this profile, t a parameter, $\alpha \in \mathbb{N}$. Typically

$$W_{t,\alpha}(x) = \left(\frac{\Lambda\mu_\alpha}{\Lambda^2\mu_\alpha^2 + \frac{|x-x_0|^2}{n(n-2)}}\right)^{\frac{n-2}{2}} ,$$

or a sum of such objects, and t represents both Λ and x_0. The goal is to find $\varphi_{t,\alpha}$ small, negligeable in front of $W_{t,\alpha}$, such that

$$I'_\alpha\left(W_{t,\alpha} + \varphi_{t,\alpha}\right) = 0$$

if we denote by I_α the functional associated to the equations. In rough approximation,

$$I'_\alpha(W_{t,\alpha}+\varphi_{t,\alpha}) = I'_\alpha(W_{t,\alpha}) + I''_\alpha(W_{t,\alpha}).(\varphi_{t,\alpha}) + \text{l.o.t.}\ ,$$

where, by l.o.t., we mean lower order terms that we are, in this rough presentation, going to neglict. Typically, in the Euclidean model case, $I_\alpha = I$ and

$$I''_\alpha(W_{t,\alpha}).(\varphi,\psi) = \int (\nabla\varphi\nabla\psi) - (2^\star - 1)\int W_{t,\alpha}^{2^\star-1}\varphi\psi\ ,$$

and in the historical model studied by Rey (1990) $-\varepsilon_\alpha \int u^2$ was added to the functional. Suppose $I''_\alpha(W_{t,\alpha}) \in \mathcal{L}(H^1,(H^1)^\star)$ has no kernel. The operator involved in $I''_\alpha(W_{t,\alpha})$ is like $T : H^1 \to H^1$ given by

$$T\varphi = \varphi - \Delta^{-1}\left((2^\star-1)W_{t,\alpha}^{2^\star-2}\varphi\right)$$

and thus of the form $Id - K$, where K is compact. By the Freedholm theory this means that $I''_\alpha(W_{t,\alpha})$ is invertible and we can find a solution $\varphi_{t,\alpha}$ to our problem. If not the case, then $I''_\alpha(W_{t,\alpha})$ has a kernel. Now we suppose, and this is a key point, that the kernel consists precisely of the derivatives $\frac{\partial W_{t,\alpha}}{\partial t}$ of the profile w.r.t. the parameter (this is exactly what the Bianchi-Egnell result says for our profiles in the Euclidean model). We can solve our problem up to this kernel and thus, since we are working in H^1, we can solve the idealized equation

$$I'_\alpha(W_{t,\alpha} + \varphi_{t,\alpha}) \equiv \lambda_\alpha \Delta \frac{\partial W_{t,\alpha}}{\partial t} \ .$$

Suppose

$$\int |\nabla \frac{\partial W_{t,\alpha}}{\partial t}|^2 = \gamma_\alpha \geq \gamma_0 > 0 \ \text{ and } \ \int \nabla \frac{\partial W_{t,\alpha}}{\partial t} \nabla \frac{\partial \varphi_{t,\alpha}}{\partial t} = o(1)$$

(the latest since $\varphi_{t,\alpha}$ is small). Define Φ_α by

$$\Phi_\alpha(t) = I_\alpha(W_{t,\alpha} + \varphi_{t,\alpha})$$

(often referred to as the reduced functional). Then

$$\begin{aligned} \Phi'_\alpha(t) &= I'_\alpha(W_{t,\alpha} + \varphi_{t,\alpha}).(\frac{\partial W_{t,\alpha}}{\partial t} + \frac{\partial \varphi_{t,\alpha}}{\partial t}) \\ &= \lambda_\alpha \gamma_\alpha + o(1) \end{aligned}$$

so that $\Phi'_\alpha(t) = 0 \ \Leftrightarrow \ \lambda_\alpha = 0$ and we get a solution to our problem if such a critical point exists. In general,

$$\Phi_\alpha(t) = I_\alpha(W_{t,\alpha}) \ + \ \text{l.o.t.}$$

and we are back to Aubin-Schoen type test functions computations to get an expression of the reduced functional from which we hope to extract a critical point.

IV.3) Extreme potentials :

The results discussed in IV.1 all involve (apart in dimension 3, or in the Chen-Wei-Yan unbounded case) sequences $(h_\alpha)_\alpha$ of potentials which converge to the Yamabe potential $\frac{n-2}{4(n-1)}S_g$. The questions we ask here are : **(i)** can we have a limit potential which equals the Yamabe potential at only one point, and, **(ii)** on the other side, what happens if not only the limit potential but all the h_α's are equal to the Yamabe potential ?

The following theorem answers the first question.

Theorem : (Nontrivial potentials, H.-Vaugon, 2001)

Let (M,g) closed, $n \geq 4$, $x_0 \in M$, and g be such that $W_g \equiv 0$ around x_0. There exists a conformal metric $\tilde{g} \in [g]$ such that $S_{\tilde{g}}$ is maximal at x_0 and only at x_0, there exists a sequence $(h_\alpha)_\alpha$ converging smoothly to $\frac{n-2}{4(n-1)}S_{\tilde{g}}(x_0)$, and there exists a sequence $(u_\alpha)_\alpha$ of smooth solutions of (E_{h_α}) w.r.t. $\tilde{g}$ which blows up in a $B_\alpha + R_\alpha$ configuration with x_0 as geometric blow-up point (one bubble, x_0 as geometric blow-up point).

In particular, the limit potential for blow-up may equal the Yamabe potential at only one point. The proof of the result is based on the notions of weakly critical and critical potentials for sharp Sobolev

The answer to the second question is given by the following remarkable result of Brendle and Brendle-Marques. The result is also based on the use of the Lyapounov-Schmidt method.

Theorem : (A counter example to the compactness conjecture, Brendle, 2008, Brendle-Marques, 2009)

Let S^n be the n-sphere, $n \geq 25$. There exists a nonconformally flat metric $\tilde{g}$ in S^n and a sequence $(u_\alpha)_\alpha$ of solutions of $(E_{\tilde{h}})$, $\tilde{h} \equiv \frac{n-2}{4(n-1)} S_{\tilde{g}}$, which blows up in a $B_\alpha + R_\alpha$ configuration (one bubble, one geometric blow-up point).

The metric $\tilde{g}$ in this result is chosen to be close to the standard metric of the sphere.

All the results we described up to now are results where we contradict compactness (blowing-up sequences of solutions of a fixed equation), analytic stability (bounded energy sequences of solutions of perturbed equations which blow-up), or bounded stability (unbounded energy sequences of solutions of perturbed equations).

IV.4) Stable situations - Bounded stability :

Now we discuss a priori analysis and want to obtain stability results for our model equation. The goal in this section is get bounded stability for our model equation. The method in order to prove bounded stability goes back to the seminal work of Schoen on the Yamabe equation. Given a converging sequence $(h_\alpha)_\alpha$ of potentials, and a sequence $(u_\alpha)_\alpha$ of solutions of

$$\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{2^\star - 1} , \qquad (E_\alpha)$$

the idea is to modelise what a blow-up point would be (very roughly speaking a critical point for u_α, kind of local maximum, at which u_α goes to $+\infty$), to develop a priori estimates around this hypothetical blow-up point like if it was alone, to prove that it is indeed the case that it is alone (isolated), and then contradict its existence by dealing with $B_\alpha + R_\alpha$ configurations (one bubble, one geometric blow-up point). In the process, by proving that blow-up points are isolated, we prove that the u_α's are actually bounded in H^1.

There are two main results which have been proved when dealing with bounded stability. The first one (stated with our stability terminology) is as follows.

Theorem : (Small potentials, Li-Zhu ($n = 3$), 1999 ; Druet ($n \geq 4$), 2004)

Let (M, g) closed, $n \geq 3$, and $h \in C^1$ be such that $\Delta_g + h$ is coercive. Assume

$$h < \frac{n-2}{4(n-1)} S_g \tag{H1}$$

everywhere in M. Then (E_h) is C^1-bounded and stable.

In the case of the Yamabe equation (dealing with compactness), the following result holds true.

Theorem : (Compactness for the Yamabe equation, Schoen, 1991 ; Khuri-Marques-Schoen, 2009)

Let (M, g) closed, $n \geq 3$, not conformally diffeomorphic to the n-sphere. Then the Yamabe equation

$$\Delta_g u + \frac{n-2}{4(n-1)} S_g u = u^{2^\star - 1} \tag{Y}$$

is bounded and compact either when g is conformally flat (and n is arbitrary) or assuming $3 \leq n \leq 24$ (and that the positive mass theorem holds true) when g is nonconformally flat.

As a very general remark, the positive mass theorem holds true when $n \leq 7$ and when g is spin in case $n \geq 8$. Then the bound $n \leq 24$ matches precisely with the noncompactness result of Brendle-Marques ($n \geq 25$). Few remarks are in order.

Rk0 : Previous versions of the compactness of the Yamabe equations on nonconformally flat manifolds were by Coda-Marques for $3 \leq n \leq 7$, Druet for $3 \leq n \leq 5$ and Li-Zhang for $3 \leq n \leq 11$.

Rk1 : When $n = 3$, by the H.-Wei result in S^3 (resonant states which appear at different values above $\frac{3}{4}$ in S^3) it is necessary to assume something like $(H1)$ in the first theorem of the preceding slides when $n = 3$. By the result of Chen-Wei-Yan (existence of unbounded sequences of solutions for (E_λ) in S^n when $\lambda > \frac{n(n-2)}{4}$), it is also necessary to assume something like $(H1)$ in higher dimensions.

Rk2 : Compactness turns out to be quite different from stability. The Yamabe equation is bounded and compact on several manifolds by the above second theorem. On the other hand, by the Esposito-Pistoia-Vétois result, it is unstable when $n \geq 4$ and by the Druet-H. and Robert-Vétois results, complex blow-up configurations may occur.

IV.5) Stable situations - Analytic stability :

Bounded stability had to do with small potentials ($h \leq \frac{n-2}{4(n-1)} S_g$). We want to recover here the full range of potentials. The following result holds true.

Theorem : (Arbitrary potentials, Druet, 2003)

Let (M, g) closed, $n \geq 4$, and $h \in C^1$ be such that $\Delta_g + h$ is coercive. Assume

$$h \neq \frac{n-2}{4(n-1)} S_g \qquad (H2)$$

everywhere in M. Then (E_h) is C^1-analytically stable when $n \neq 6$.

Rks : (i) $(H2)$ is a very natural relaxation of $(H1)$. **(ii)** The result does not require that S_g should be positive (positive Yamabe invariant). In particular we get analytic stability in nonpositively curved manifolds if, for instance, we assume $h > 0$. **(iii)** By the blow-up examples we discussed above, $(H2)$ is a necessary assumption. **(iv)** The original result of Druet required a C^2-convergence of the potentials. The C^2-convergence was later on relaxed to a C^1-convergence (H.-Druet, 2009). **(v)** In dimensions $3, 4, 5$, (Li-Zhu, Druet) $u_\infty \equiv 0$ in case of blow-up.

The proof of the theorem is based on the C^0-theory for blow-up (See Part II) and the notion of range of interactions of bubbles (due to Druet). It involves the whole machinery in the nonconformally flat case. A "lighter" version of the proof can be given on conformally flat manifolds.

Dimension 6 was a surprise in the theorem. It has been a question for some time to decide whether or not it was a purely technical artefact in the result (6 is the sole dimension for which the L^2-terms in μ_α^2 compete with the boundary terms $\mu_\alpha^{(n-2)/2}$ at the scale $\sqrt{\mu_\alpha}$). The following result answers the question.

Proposition : (The 6-dimensional case, Druet-H., 2009)

There exist $h : S^6 \to \mathbb{R}$, $h > 6$ (the RHS in $(H2)$ on S^6), a sequence $(h_\alpha)_\alpha$ of smooth functions in S^6 converging C^1 to h, and a sequence $(u_\alpha)_\alpha$ of smooth positive solutions of

$$\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{2^\star - 1}$$

for all α, such that $(u_\alpha)_\alpha$ blow up in a $u_\infty + B_\alpha + R_\alpha$ configuration with $u_\infty \not\equiv 0$ (a positive limit profile, one bubble, one geometric blow-up point).

IV.6) Extensions - Limitations :

The bounded stability theorem extends to asymptotically critical subcritical perturbations like

$$\Delta_g u_\alpha + h_\alpha u_\alpha = u_\alpha^{p_\alpha - 1} ,$$

where $h_\alpha \to h$ in C^1 (as before), $p_\alpha \leq 2^\star$ for all α and $p_\alpha \to 2^\star$ as $\alpha \to +\infty$ (contrary to $p_\alpha = 2^\star$ for all α as before). If we forget about dimension 6 which we already discussed in the above proposition, and with respect to the bounded stability theorem, and this extension of the bounded stability theorem to subcritical exponents, the analytic stability theorem has two limitations :

(1) it does not allow asymptotically critical subcritical exponents,

(2) it requires bounded energy for the u_α's.

Both limitations are necessary : by the result of Chen-Wei-Yan (existence of unbounded sequences of solutions for (E_λ) in S^n for all $\lambda > \frac{n(n-2)}{4}$), and also by a result of Micheletti-Pistoia-Vétois (2009) which states that on any closed n-manifold, $n \geq 4$, there are potentials $h > \frac{n-2}{4(n-1)} S_g$ for which the equations $\Delta_g u_\alpha + h u_\alpha = u_\alpha^{2^\star - 1 - \varepsilon_\alpha}$ possess blowing-up positive solutions $u_\alpha = B_\alpha + R_\alpha$ with $\varepsilon_\alpha \to 0$.

Both the bounded stability theorem and the analytic stability theorem extend to systems of equations (Druet-H., 2009, and Druet-H.-Vétois, 2010).

IV.7) A summary of the results we discussed :

<table>
<tr><th>Bounded stability</th><th>Analytic stability</th></tr>
<tr><td>(M,g), $n \geq 3$, $Y_g > 0$,
$h < \frac{n-2}{4(n-1)} S_g$
(Li-Zhu, $n = 3$, 1999; Druet, $n \geq 4$, 2004; see also Druet-Hebey-Vétois, 2010)</td><td rowspan="3">(M,g), $n \geq 4$
$\Delta_g + h$ coercive, $n \neq 6$,
$h \neq \frac{n-2}{4(n-1)} S_g$
(Druet, 2003; see also Druet-Hebey, 2009)</td></tr>
<tr><td>Compactness</td></tr>
<tr><td>(M,g) conf. flat or $3 \leq n \leq 24$,
$Y_g > 0$, $(M,g) \neq (S^n, g_0)$,
$h_\alpha \equiv \frac{n-2}{4(n-1)} S_g$
(Schoen, 1991; Khuri-Marques-Schoen, 2009)</td></tr>
</table>

$(B_\alpha + R_\alpha)$-config.	$(\sum B^i_\alpha + R_\alpha)$-config.	$(u_\infty + \sum B^i_\alpha + R_\alpha)$-config.	Unbounded Energy
(S^n, g_0), $n \geq 3$ $h_\alpha \equiv \frac{n(n-2)}{4}$ (Historical)	$(S^n/G, g_0)$, $n \geq 6$ $h_\alpha \overset{C^1}{\to} \frac{n(n-2)}{4}$ (Druet-Hebey, 2004)	(S^6, g_0), $h > 6$ $h_\alpha \overset{C^1}{\to} h$, 1-Bubble, $u_\infty \not\equiv 0$ (Druet-Hebey, 2009)	$(S^n/G, g_0)$, $n \geq 6$, $h_\alpha \to \frac{n(n-2)}{4}$ (Druet-Hebey, 2004)
(M, g), $n \geq 4$, $Y_g > 0$, $h_\alpha \overset{C^\infty}{\to} \frac{n-2}{4(n-1)} S_g$ (Esposito-Pistoia -Vétois, 2013)	(M, g), $n \geq 6$, non conf. flat, $Y_g > 0$, $h_\alpha \overset{C^r}{\to} \frac{n-2}{4(n-1)} S_g$ (Robert-Vétois, 2013)		(S^n, g_0), $n \geq 5$, $h_\alpha \equiv \lambda$, $\lambda > \frac{n(n-2)}{4}$ (Chen-Wei-Yan, 2011)
(S^3, g_0), $\exists (\theta_k)_k$ res. states $\theta_1 = \frac{3}{4} < \theta_k$, $k \geq 2$, $\theta_k \to +\infty$, $\lambda_\alpha \to \theta_k$ (Hebey-Wei, 2012)			(M, g), $n \geq 6$, non conf. flat, $Y_g > 0$, $h_\alpha \overset{C^r}{\to} \frac{n-2}{4(n-1)} S_g$ (Esposito-Pistoia Vétois, Robert-Vétois, 2013)
$(S^n, \tilde{g})$, $\tilde{g}$ non conf. flat, $n \geq 25$, $h_\alpha \equiv \frac{n-2}{4(n-1)} S_{\tilde{g}}$ (Brendle, 2008; Brendle-Marques, 2009)			
(M, g), $n \geq 4$, conf. flat, $h_\alpha \overset{C^\infty}{\to} \frac{n-2}{4(n-1)} \max_M S_{\tilde{g}}$ for some $\tilde{g} \in [g]$, $S_{\tilde{g}}$ max. at only one point (Hebey-Vaugon, 2001)			

Thank you for your attention !